Quantum Time-Space Tradeoff for Finding Multiple Collision Pairs

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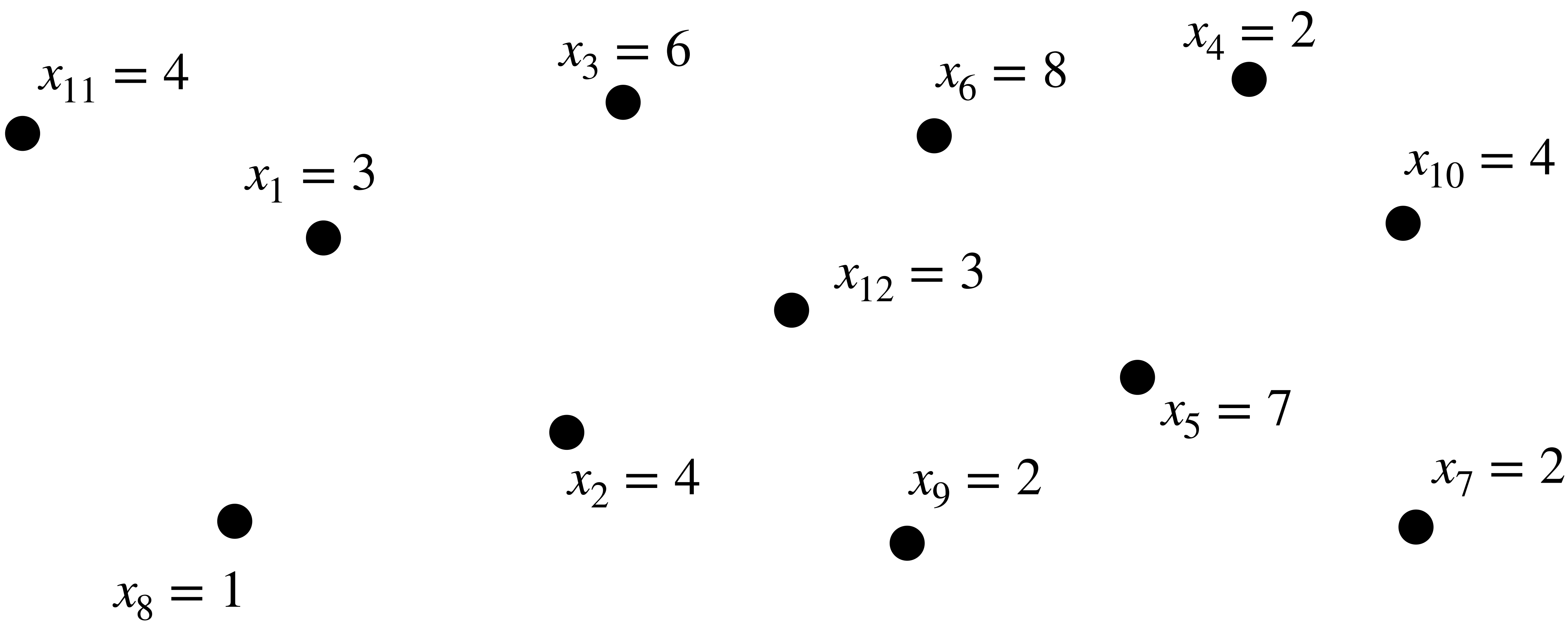
Our contribution: a new tradeoff for the **Collision Pairs Finding** problem.

1

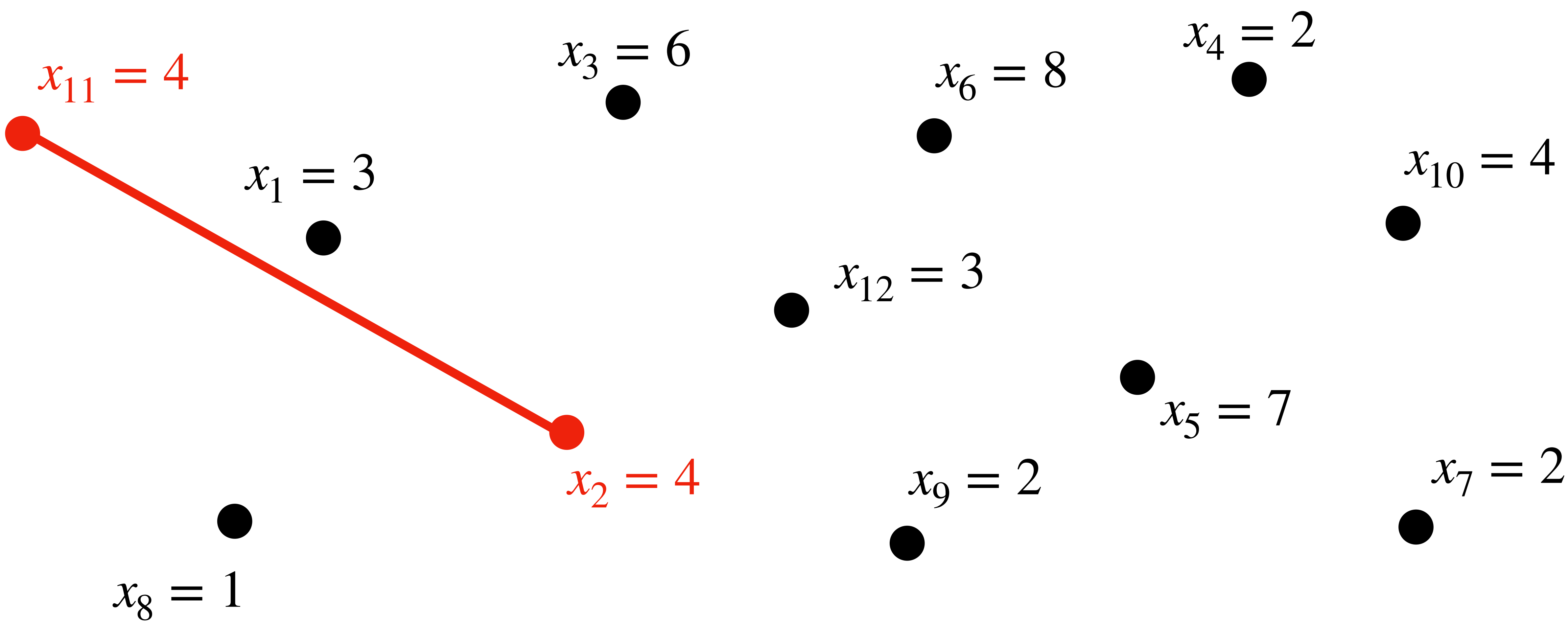
The Collision Pairs Finding Problem

Collision Pairs



Collision pair: $x_i = x_j$

Collision Pairs



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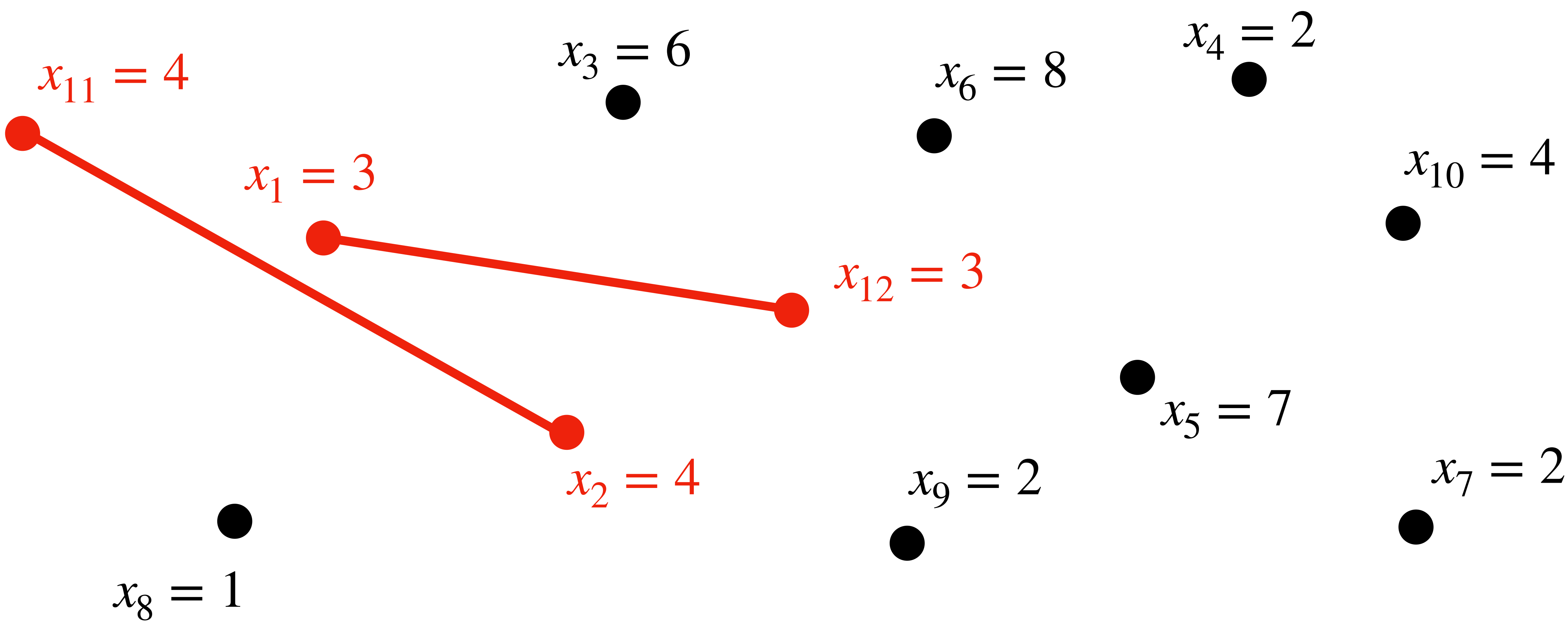

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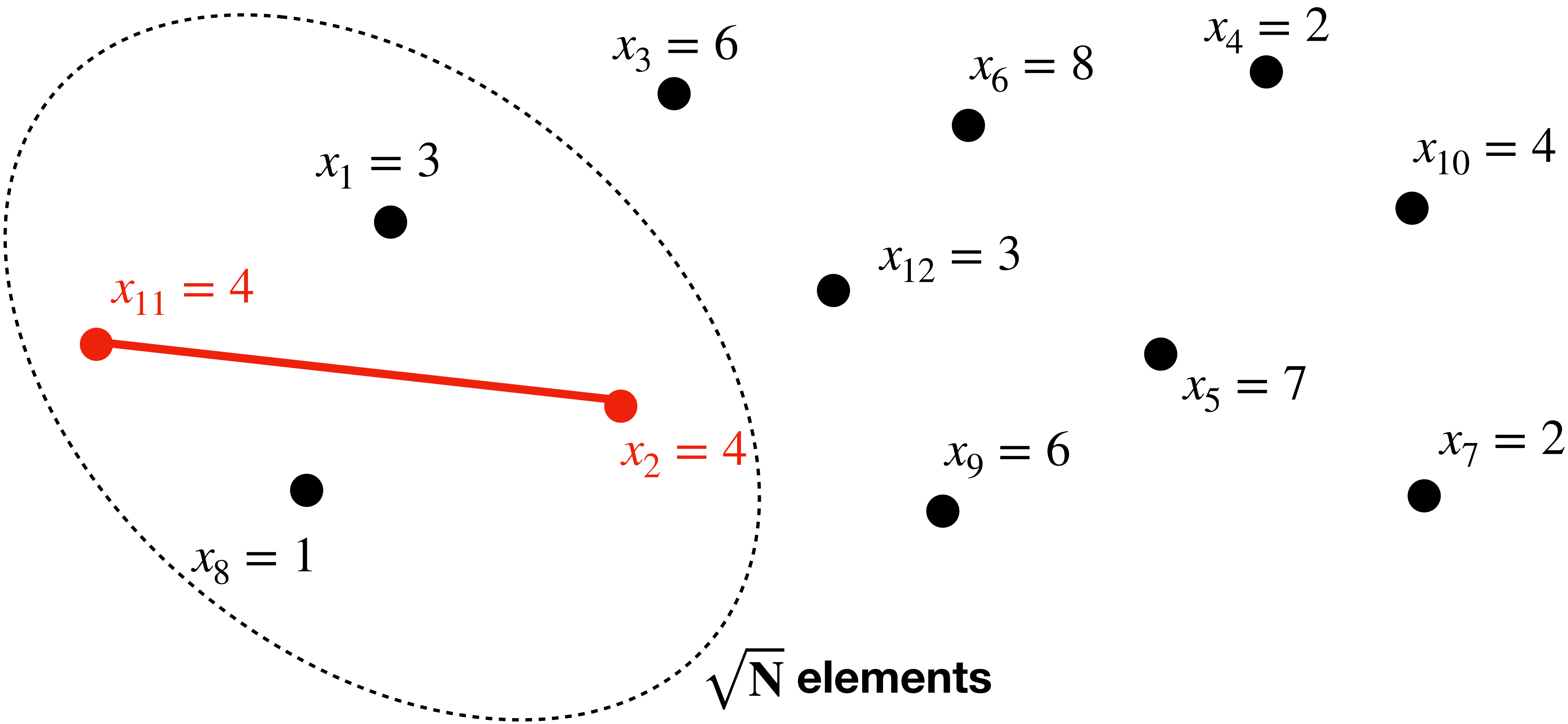
→ **Finding collisions is an important problem in cryptanalysis:**

- preimage attacks on hash functions
- meet-in-the-middle attacks ← **requires to find many collisions**
- computing discrete logarithms
- …

Birthday attack (K = 1)



Birthday attack

$$T = O(\sqrt{N})$$

$$S = O(\sqrt{N})$$

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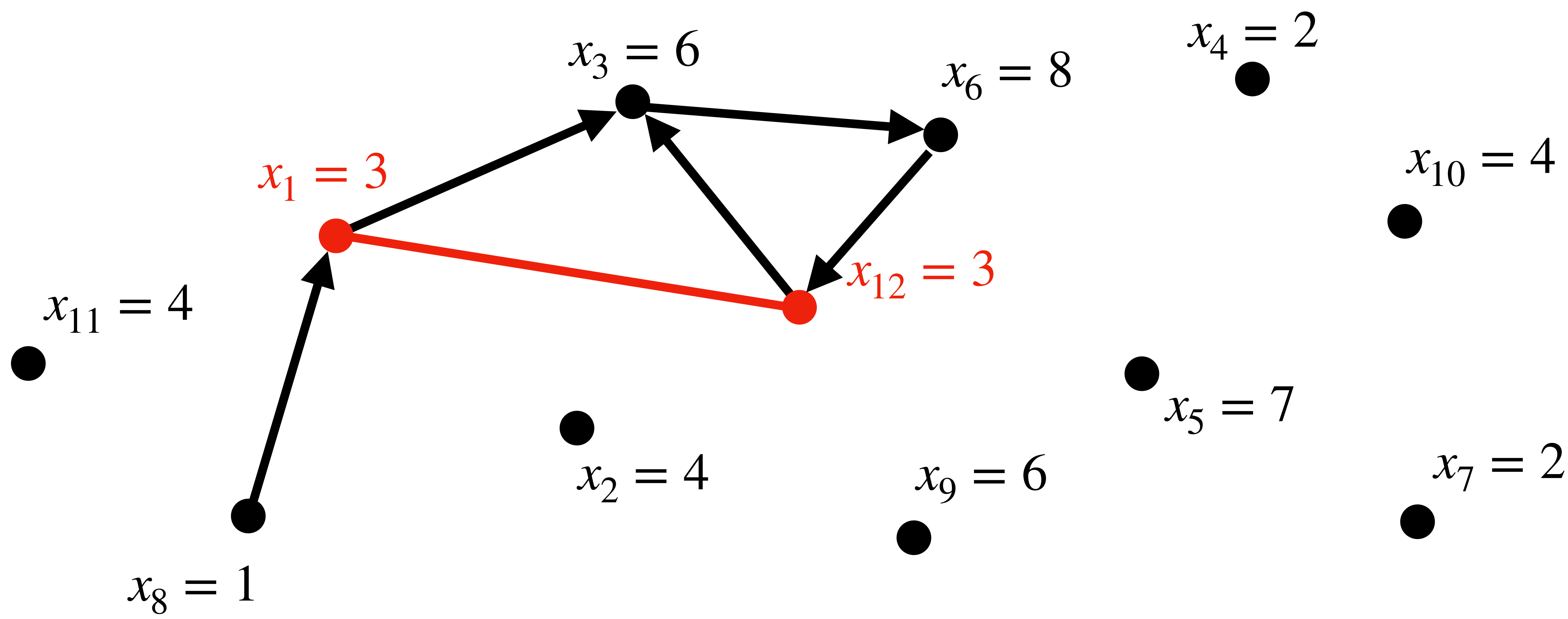


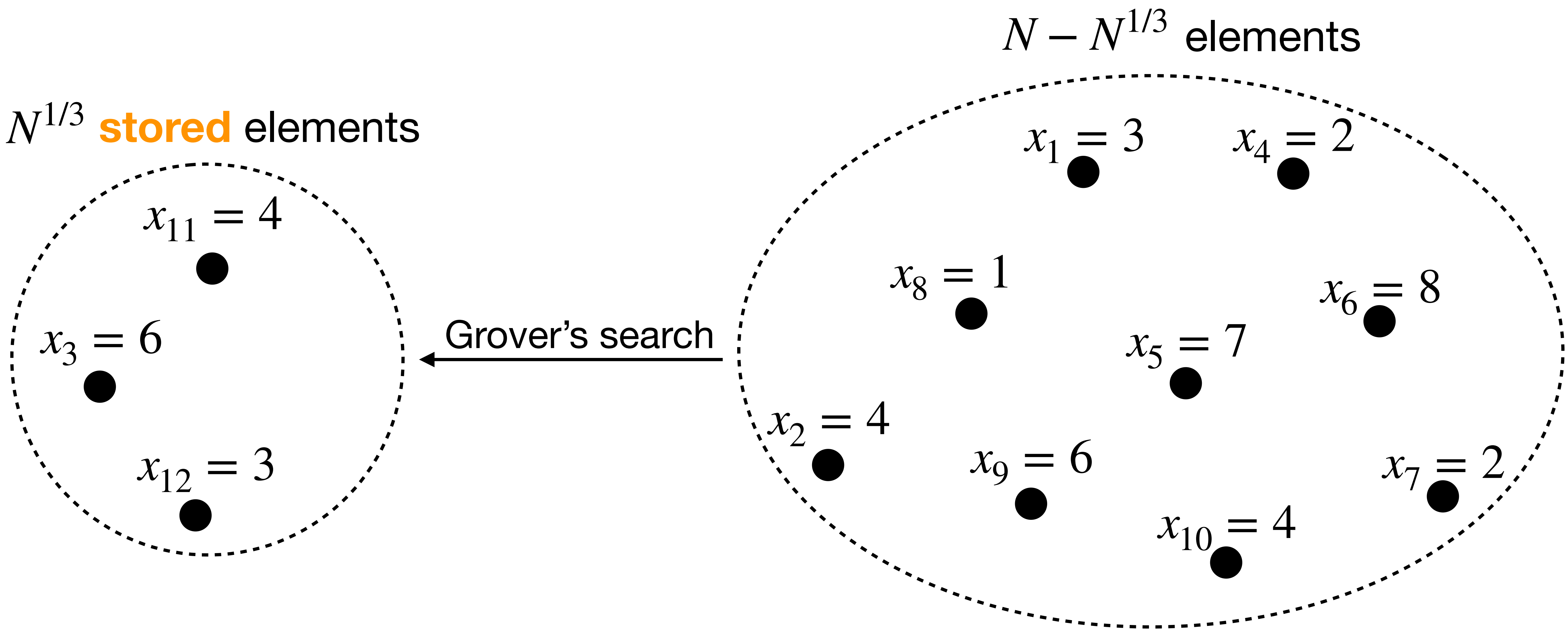
Birthday attack

$$T = O\left(\sqrt{N}\right)$$
$$S = O\left(\sqrt{N}\right)$$

Birthday attack + Floyd's cycle finding

$$T = O\left(\sqrt{N}\right)$$
$$S = O(\log N)$$

Quantum BHT algorithm (K = 1)



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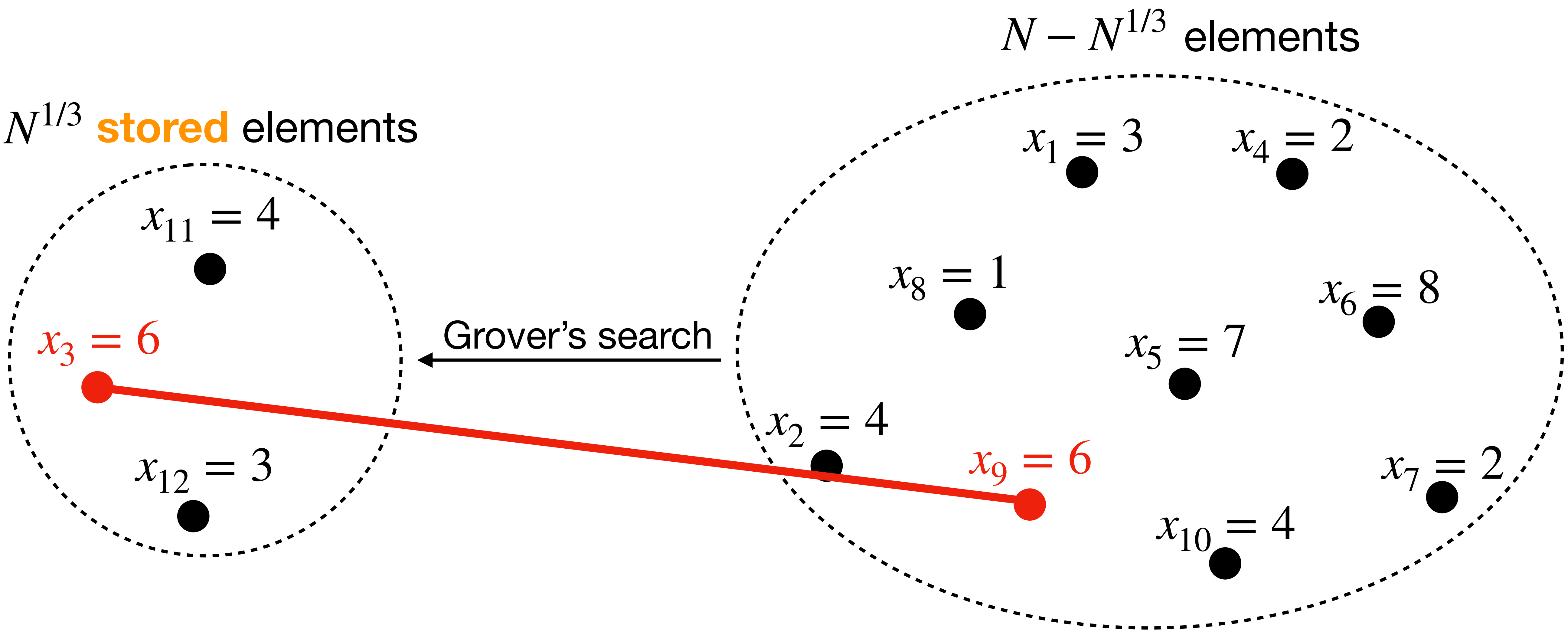

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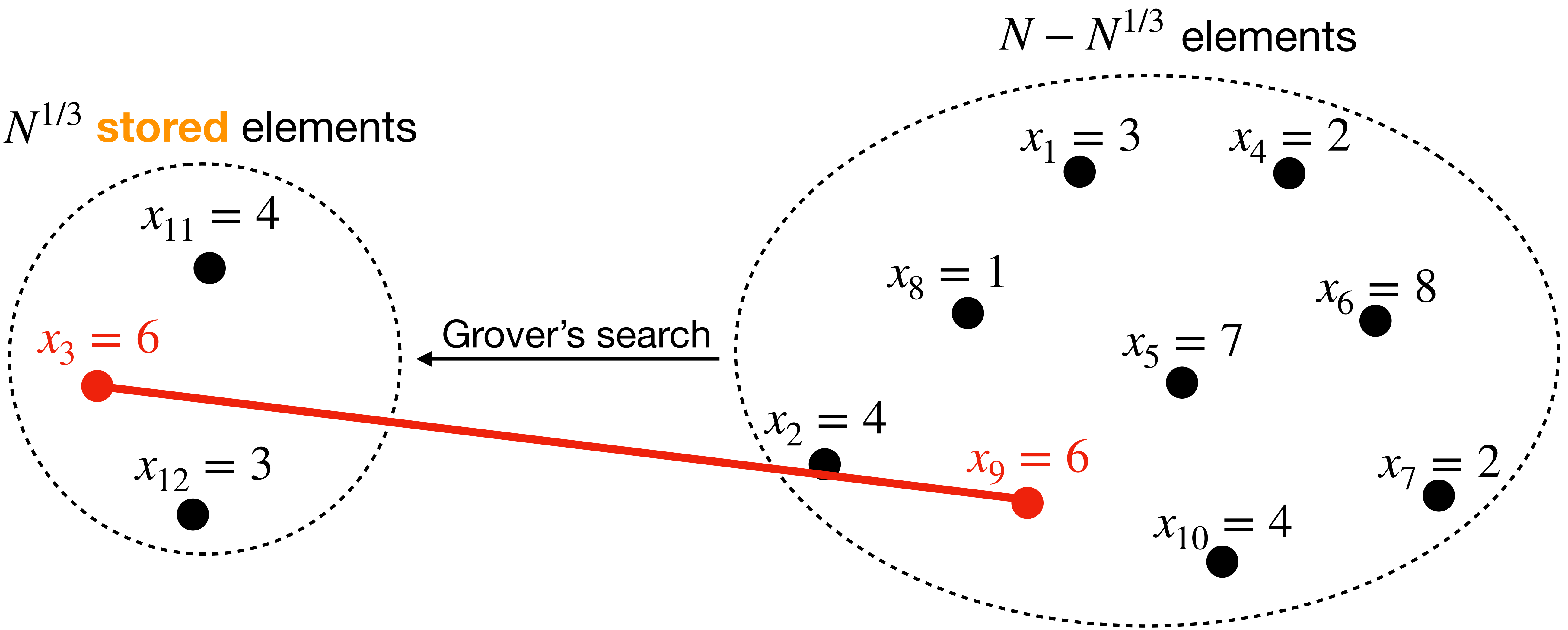

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$$T = O\left(N^{1/3}\right)$$

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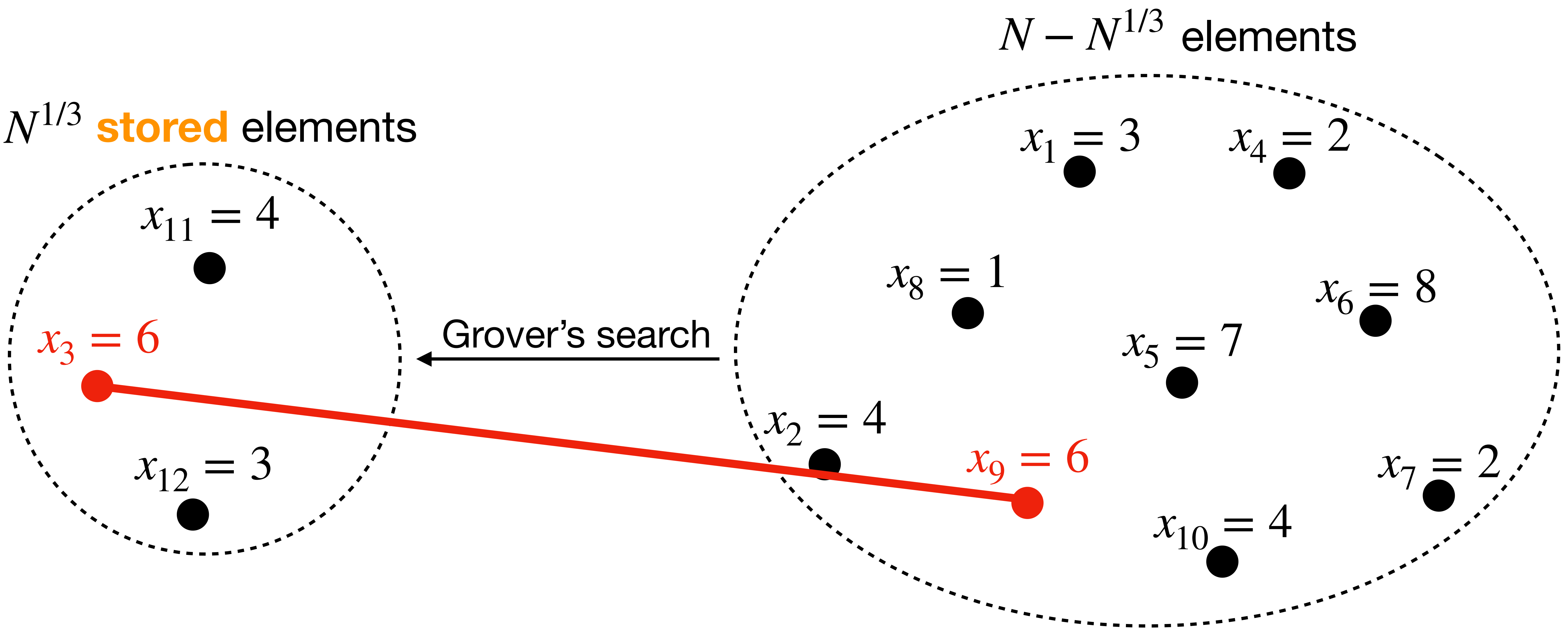

Quantum BHT algorithm

$$T = O\left(N^{1/3}\right)$$

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Open problem:

Is there a quantum algorithm with $T \leq o\left(\sqrt{N}\right)$ and $S = O(\log N)$?

Finding K collision pairs

	Classical Tradeoff
Upper bound	$T^2S \leq \tilde{O}(K^2N)$ when $\tilde{\Omega}(\log N) \leq S \leq \tilde{O}(K)$ Parallel Collision Search [van Oorschot and Wiener'99]

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Lower bound	$T^2S \geq \tilde{\Omega}(K^2N)$ [Chakrabarti,Chen'17] [Dinur'20]	$T^3S \geq \Omega(K^3N)$ Our result → $T \geq \tilde{\Omega}(KN^{1/3})$ when $S = \log(N)$

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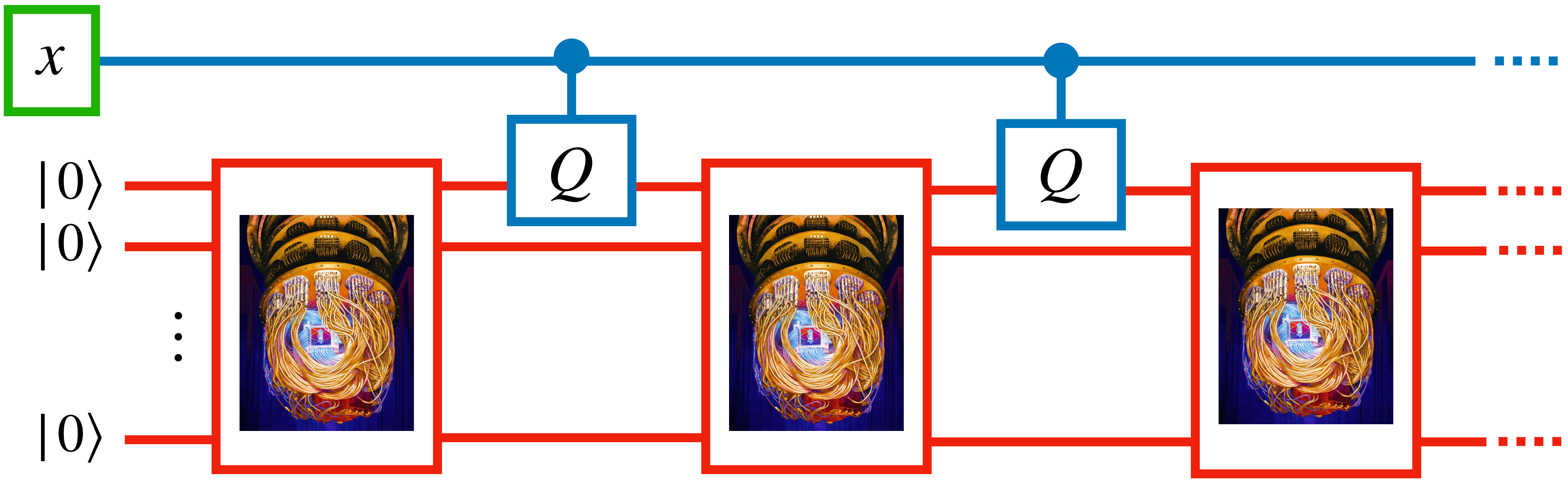
Lower Bound Method

Quantum Query Model



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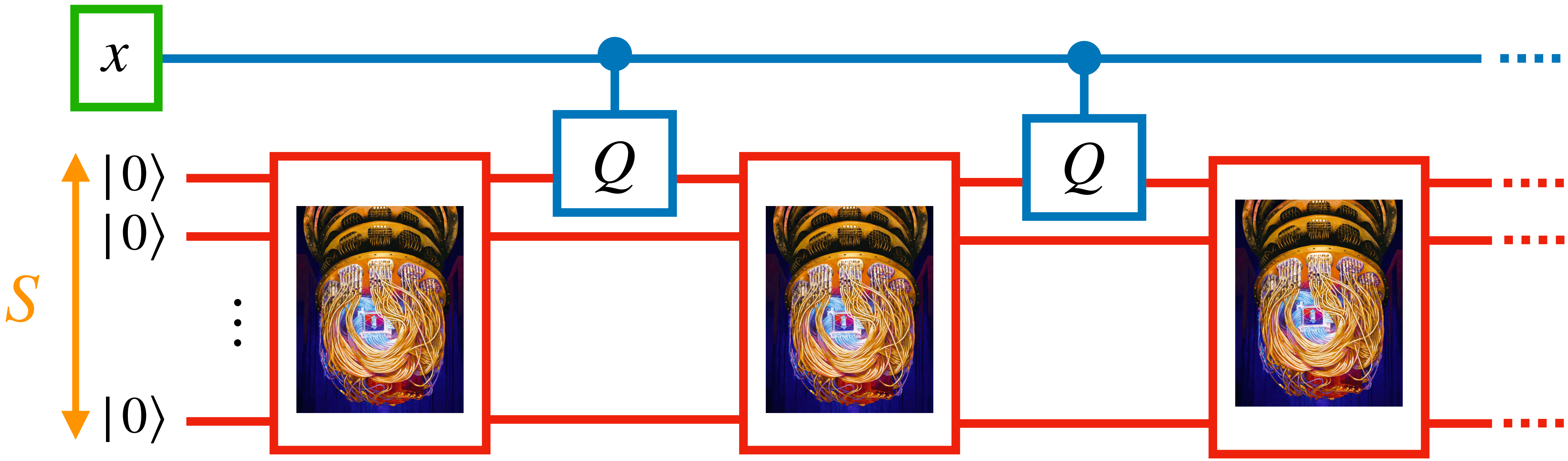

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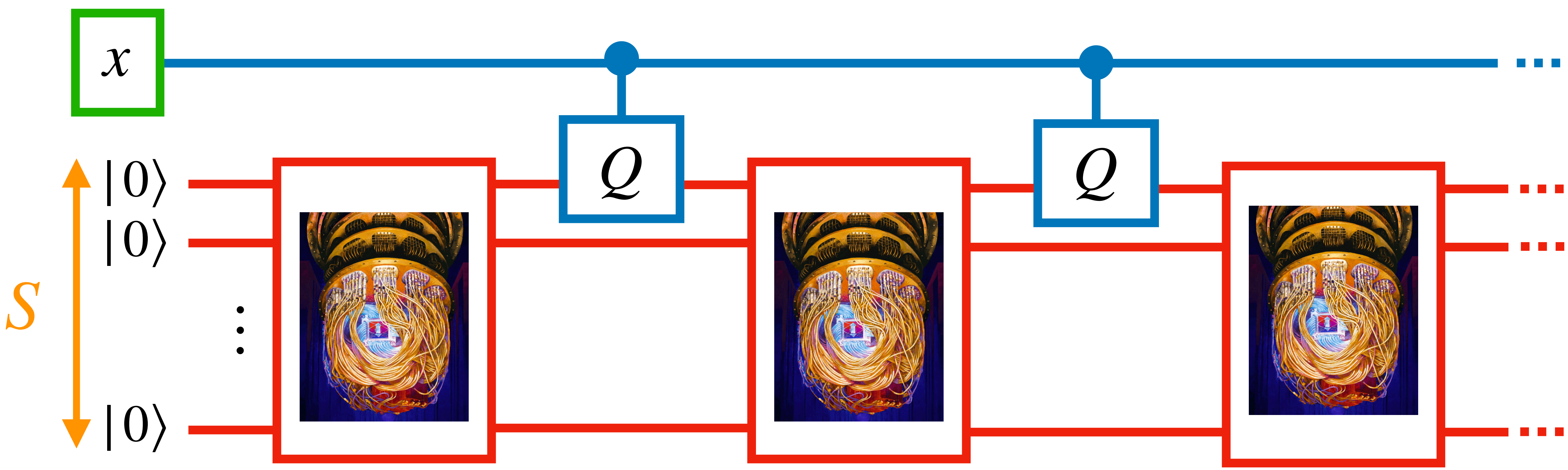

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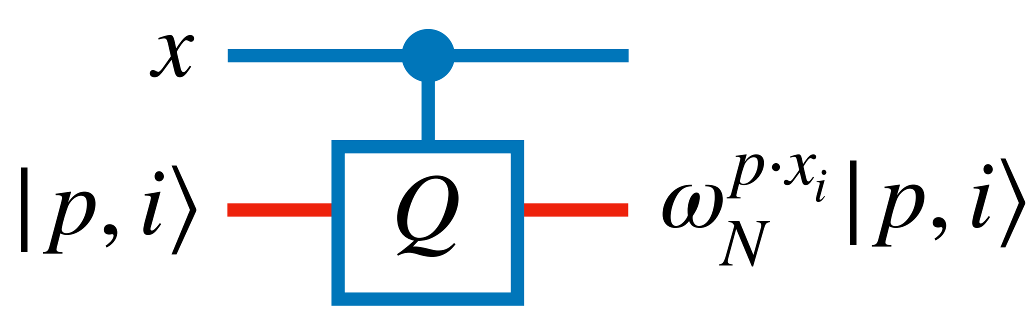

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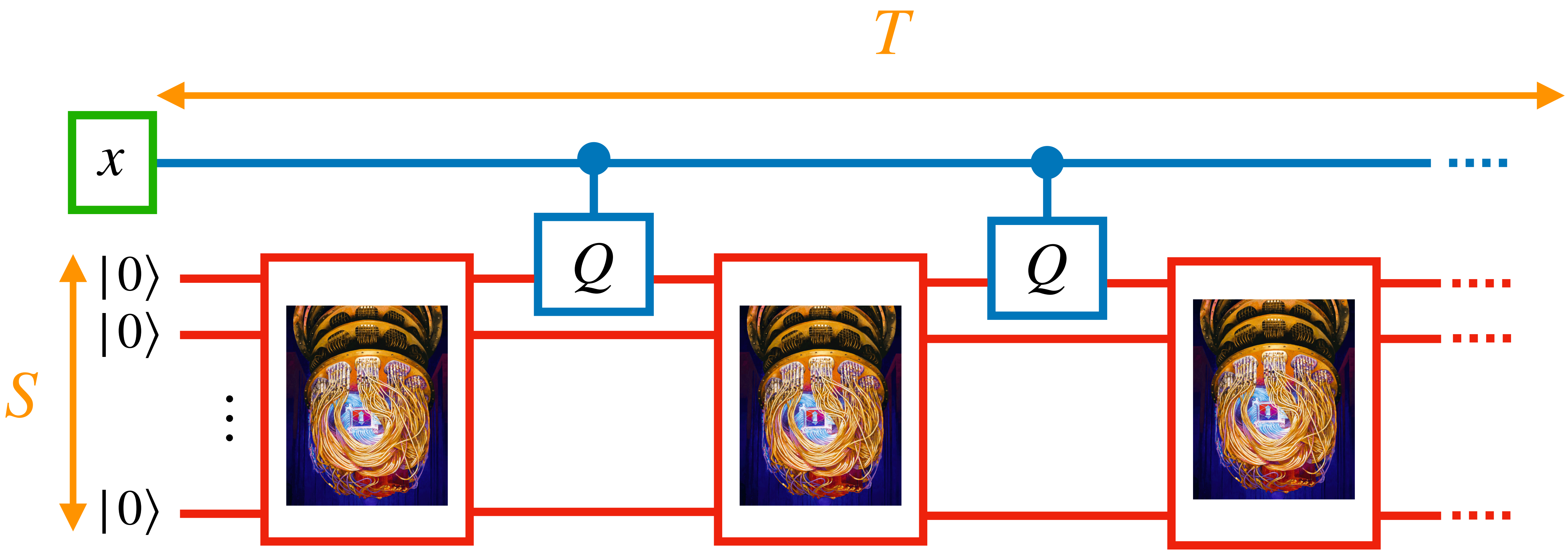

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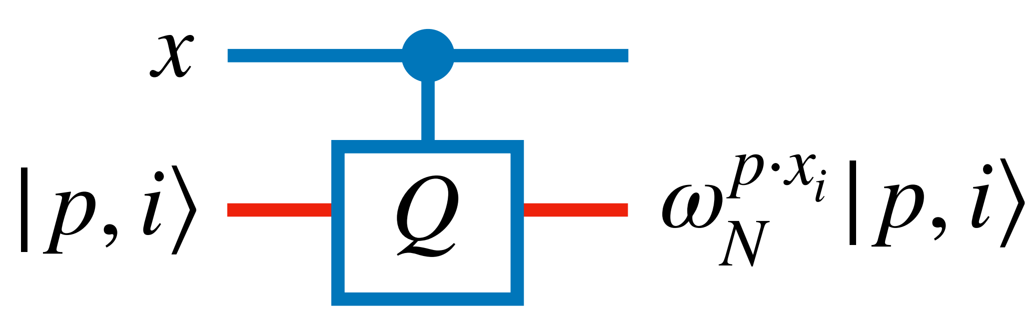
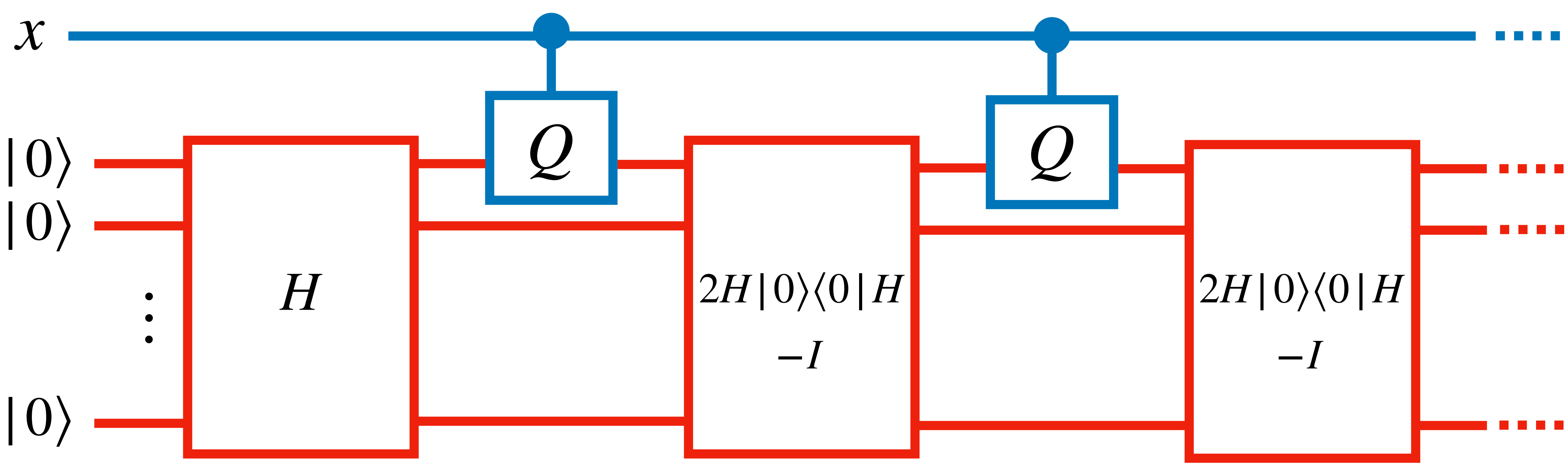


- The computation alternates between **T quantum queries** and **T quantum operations** on the memory.

Quantum Query Model



Example: Grover's Search

$$\begin{cases} T = O(\sqrt{N}) \\ S = O(\log N) \end{cases}$$

Time-Space Lower Bounds from Time Lower Bounds

- A method to deduce Time-Space lower bounds from **Time lower bounds**.
- Introduced by [Borodin et al.'81], and by [Klauck'03] for the quantum version.
- Applicable when the problem has a large **output of size K** (≠ decision problem).

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$\Longrightarrow$

$$T \geq \Omega\left(K N^{1/3} / S^{1/3}\right)$$

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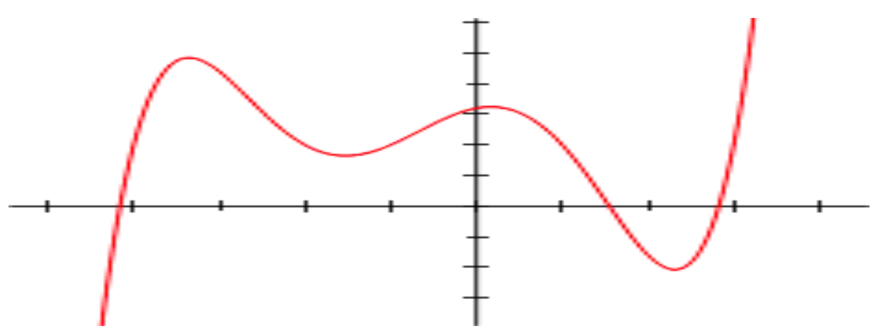
Two main methods for proving such lower bounds:

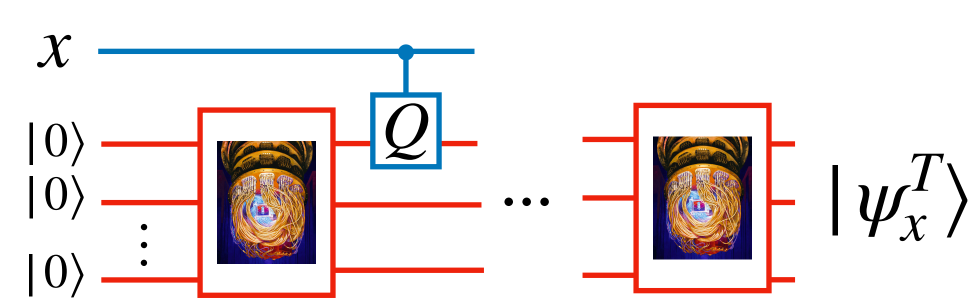
Polynomial Method

The acceptance probability of a T-query algorithm is a polynomial in x of degree at most 2T.



Adversary Method

Bound the progress $W^t = \sum_{x,y} w_{x,y} \langle \psi_x^t | \psi_y^t \rangle$.



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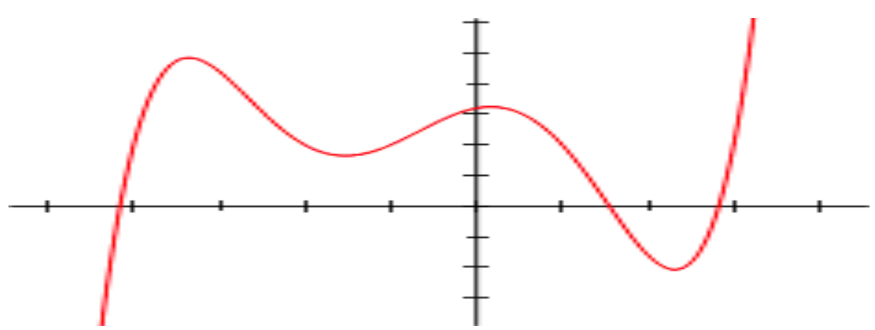

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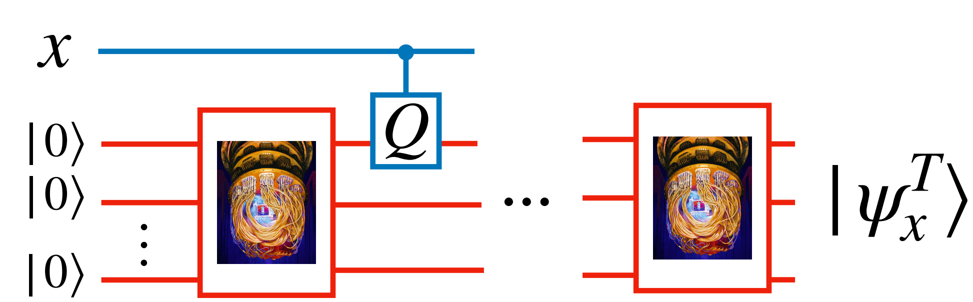

Our approach: a refined version of Zhandry's **recording technique**

Classical Recording

Input: $x = (x_1, \ldots, x_N)$ where $x_i = y \in [N]$ with probability $1/N$.

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Algorithm

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$\xrightarrow{x_2?}$

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Algorithm **Input**

$$x = (\perp, \perp, \perp, \perp)$$

$$\xrightarrow{x_2?}$$

$$x = (\perp, 4, \perp, \perp)$$

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$$\xrightarrow{x_2?}$$

$$\xleftarrow{x_2 = 4}$$

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$\xrightarrow{x_2 ?}$

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$\xrightarrow{x_1 ?}$

$\xleftarrow{x_1 = 7}$

$x = (7, 4, \perp, \perp)$

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$\xleftarrow{x_2 = 4}$ $\quad x = (7, 4, \perp, \perp)$

$\xrightarrow{x_4 ?}$

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$\vdots$

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$x = (\perp, \perp, \perp, \perp)$

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Progress measure: probability to have recorded at least **k** collisions after **t** queries

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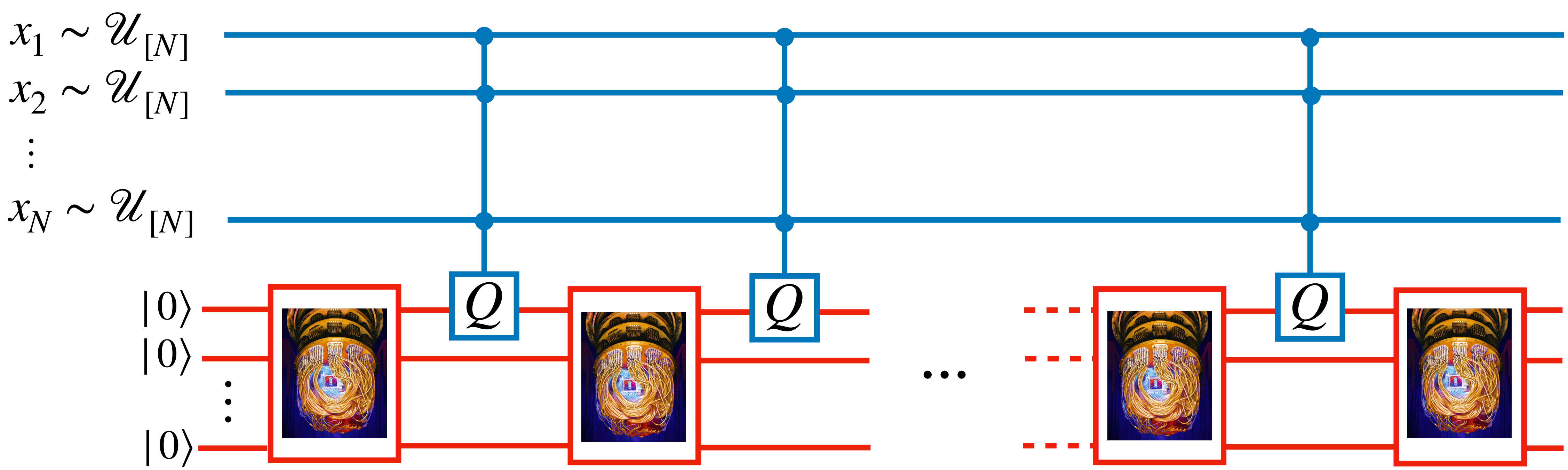
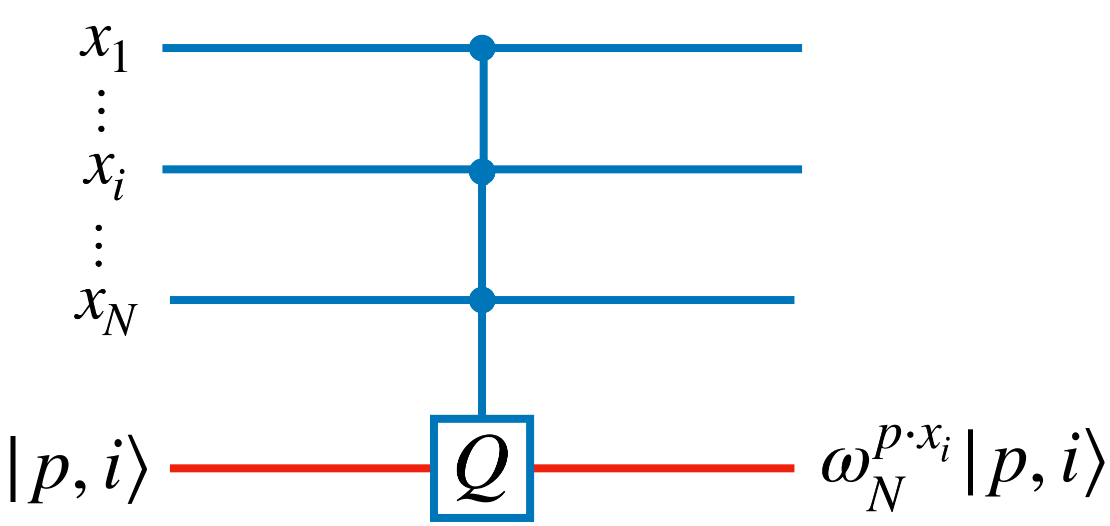
→ the algorithm must force the recording of many collisions to succeed with high probability

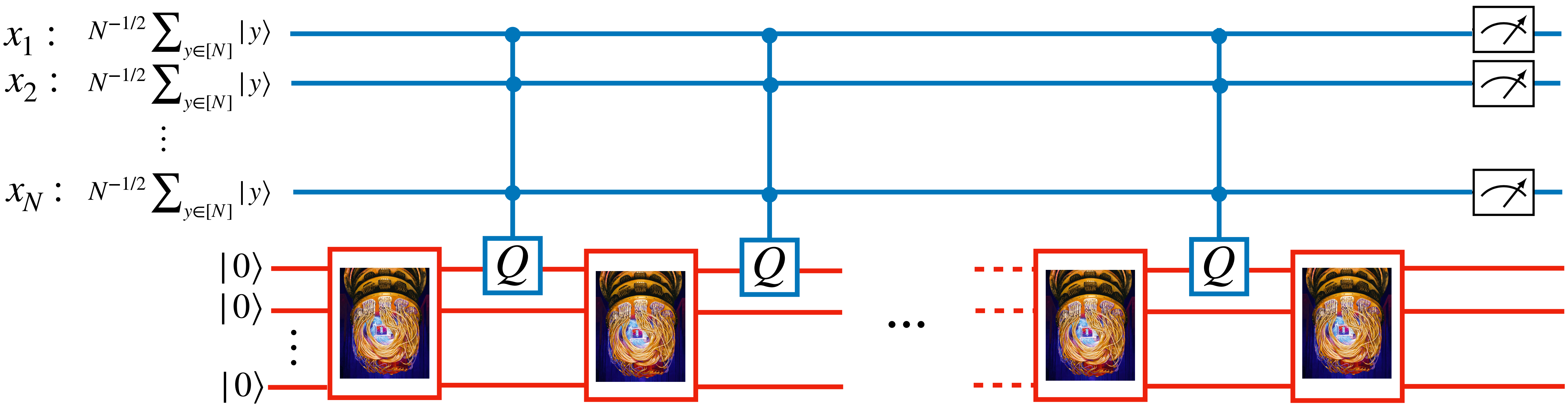
Quantum Recording



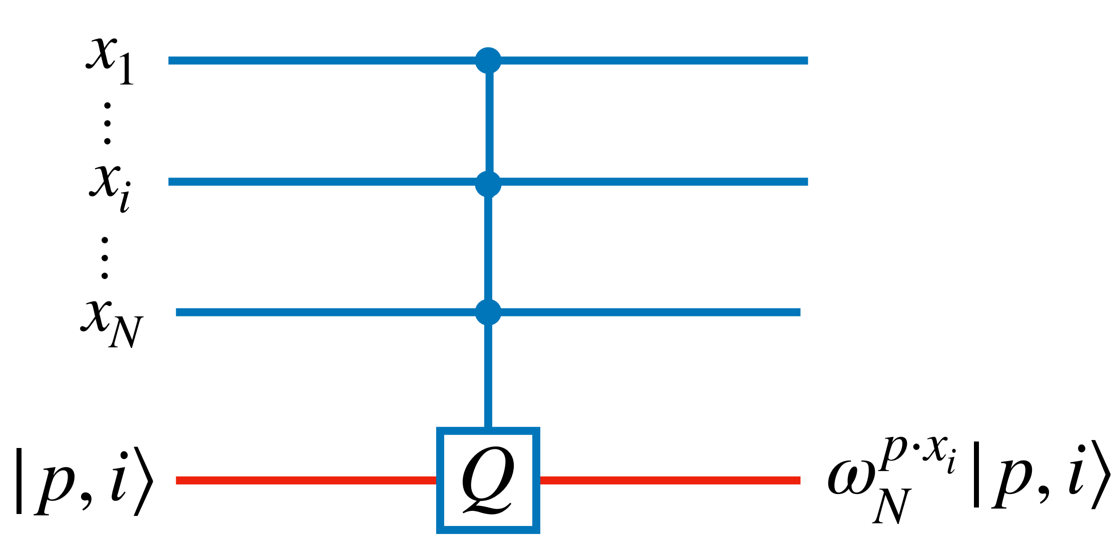
Query Operator



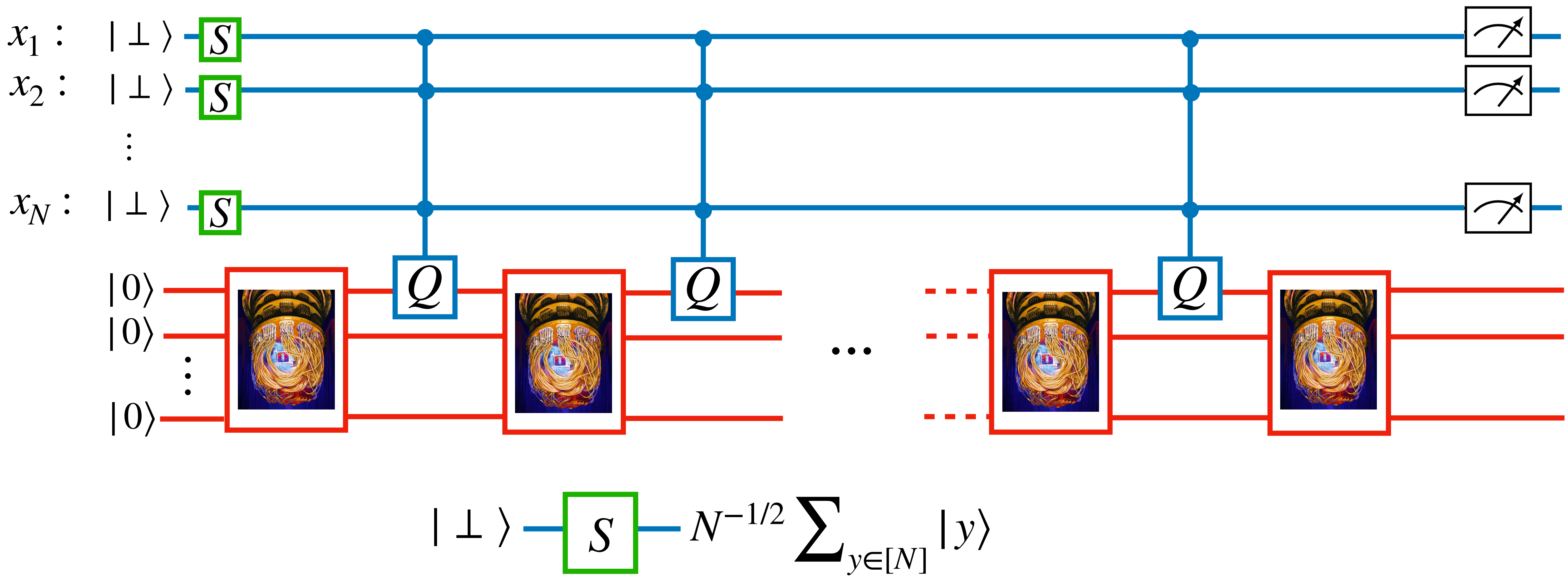
Quantum Recording



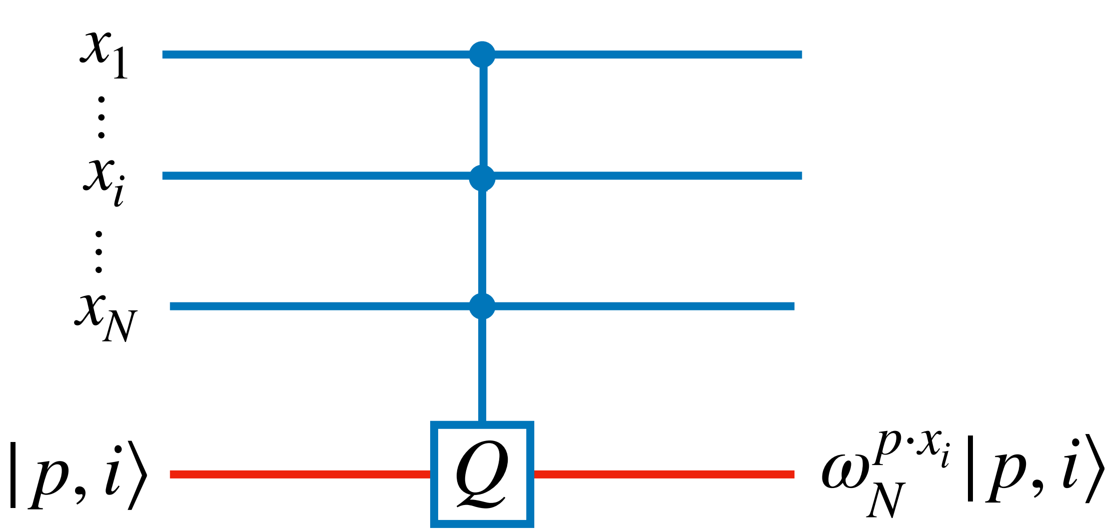
Query Operator



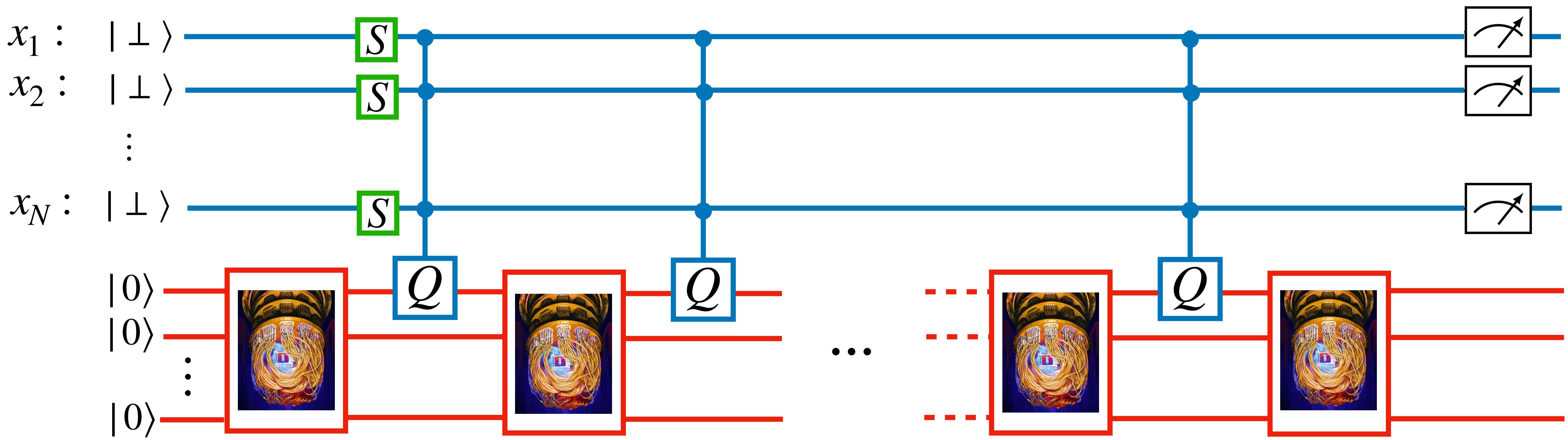
Quantum Recording

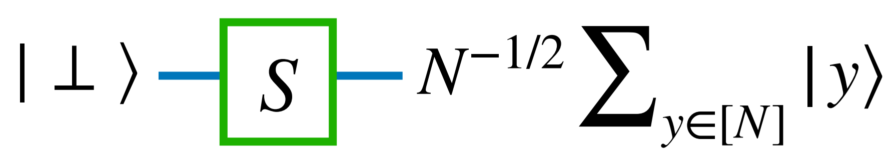


Query Operator

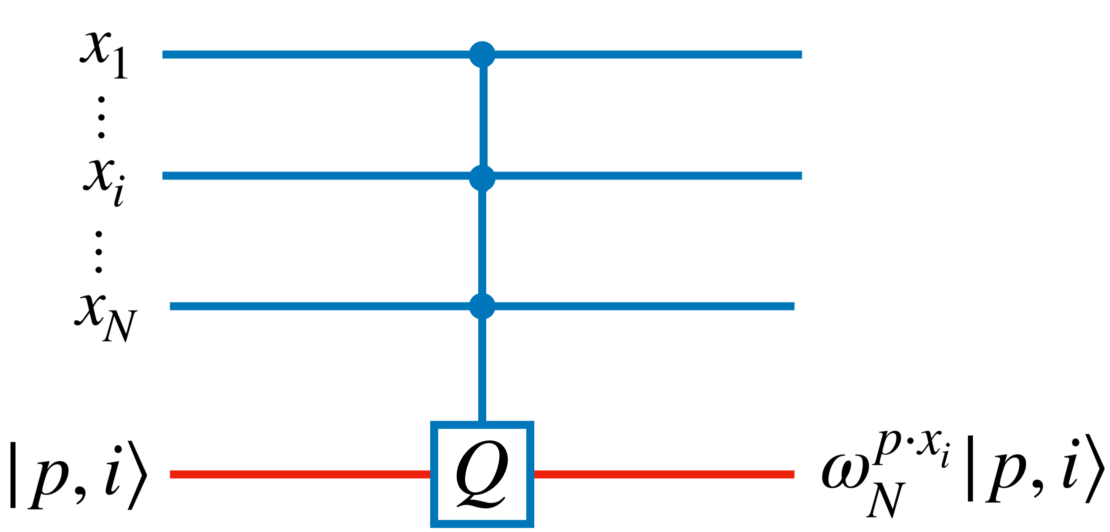


Quantum Recording

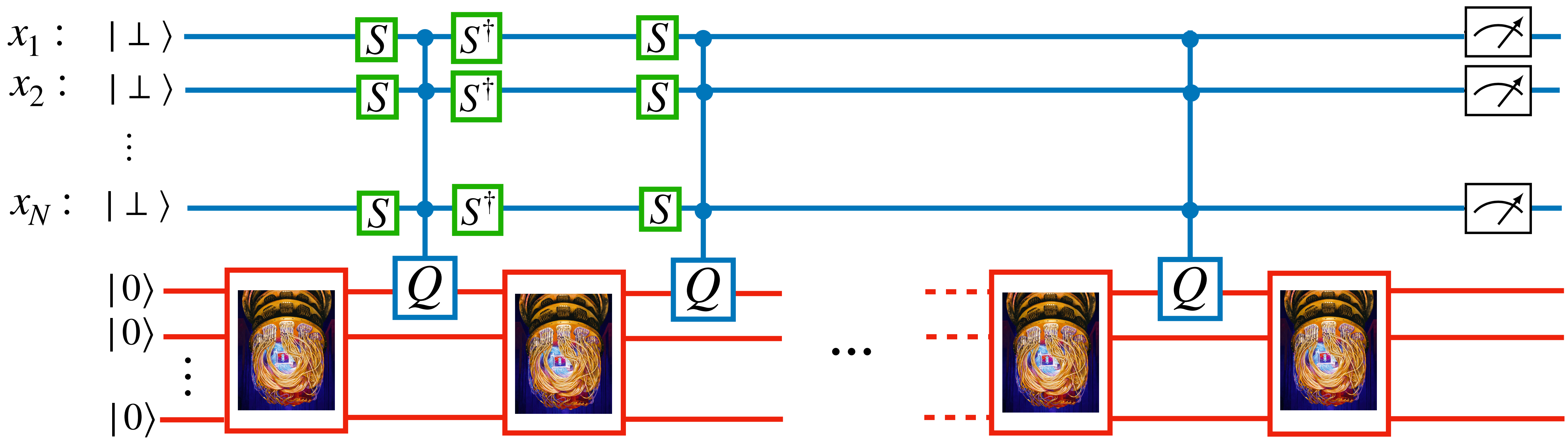


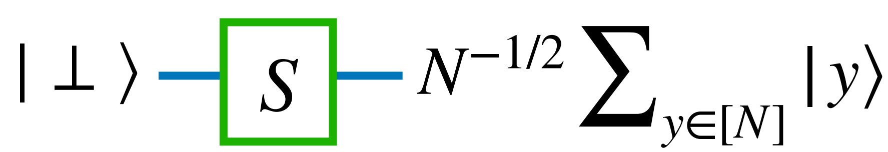


Query Operator

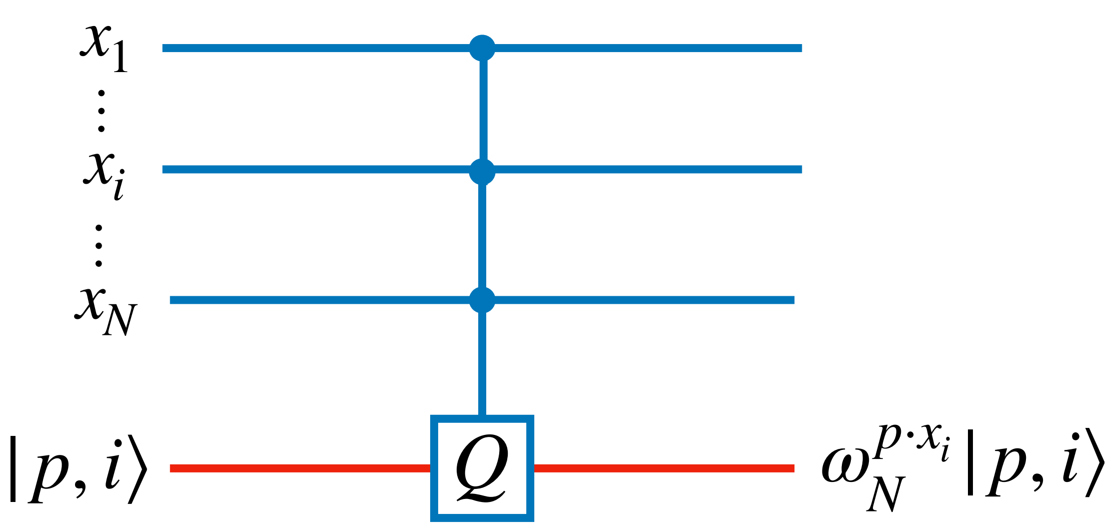


Quantum Recording

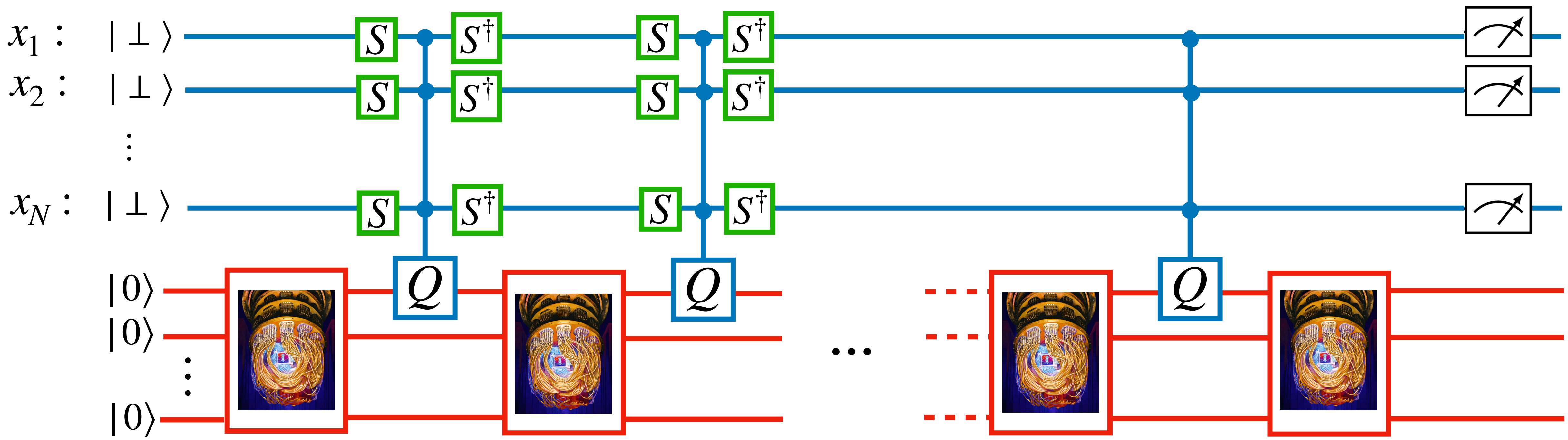


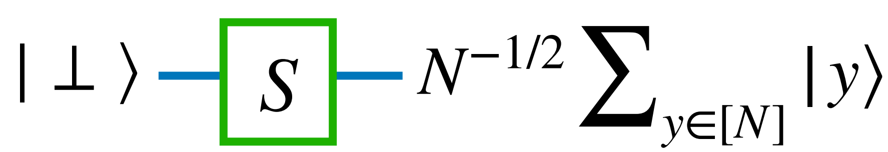


Query Operator

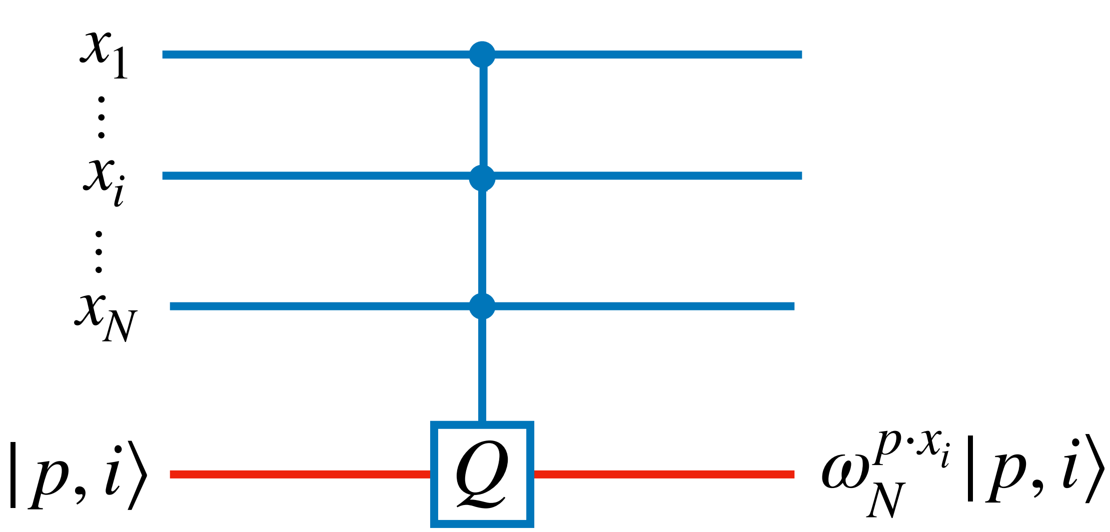


Quantum Recording

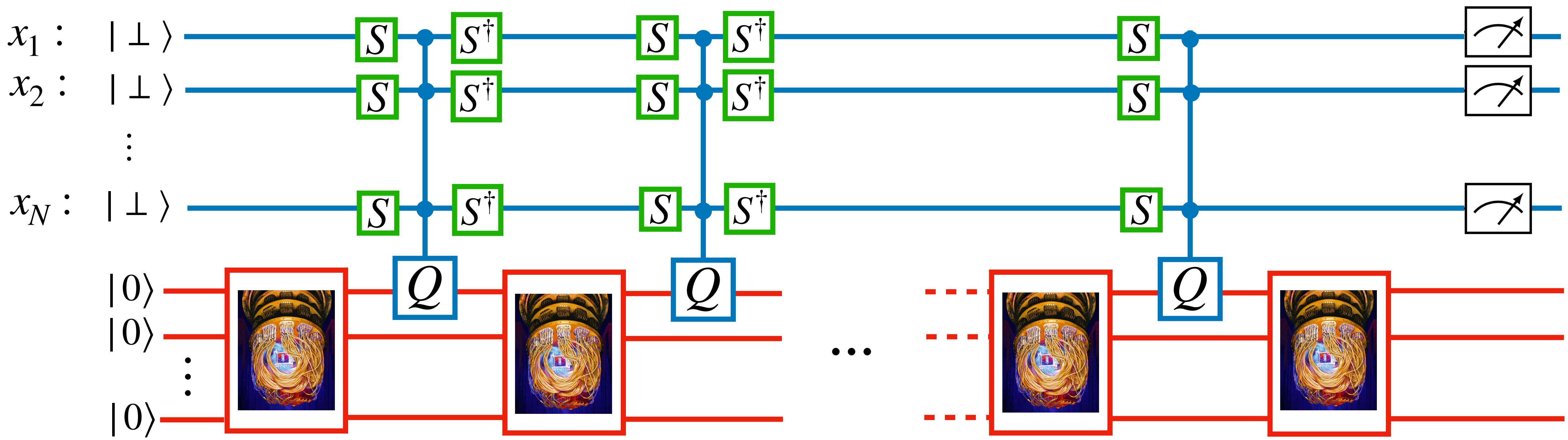




Query Operator

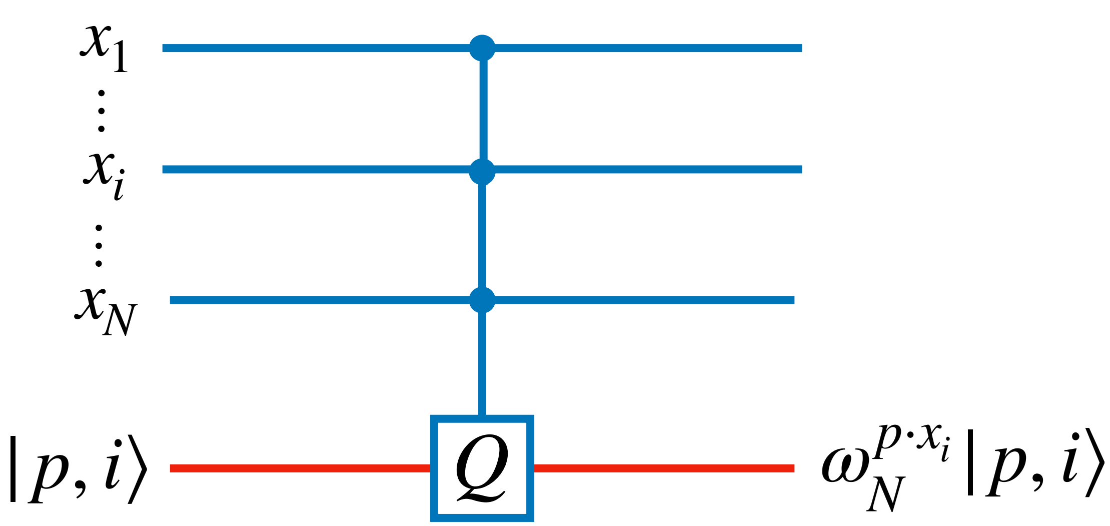


Quantum Recording

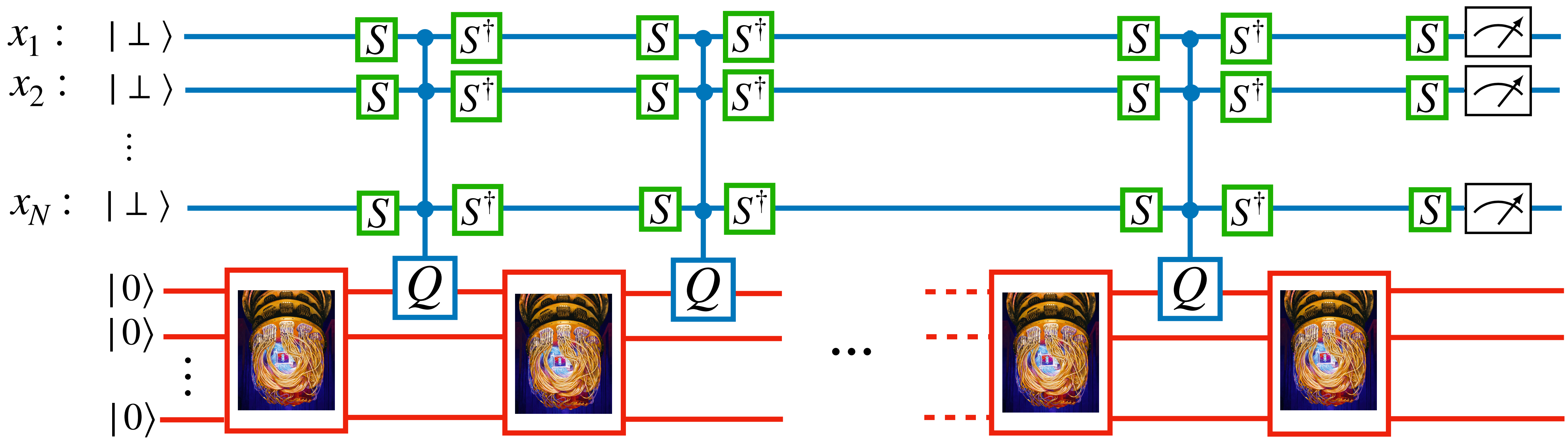


$$|\perp\rangle \; — \boxed{S} — \; N^{-1/2} \sum_{y \in [N]} |y\rangle$$

Query Operator

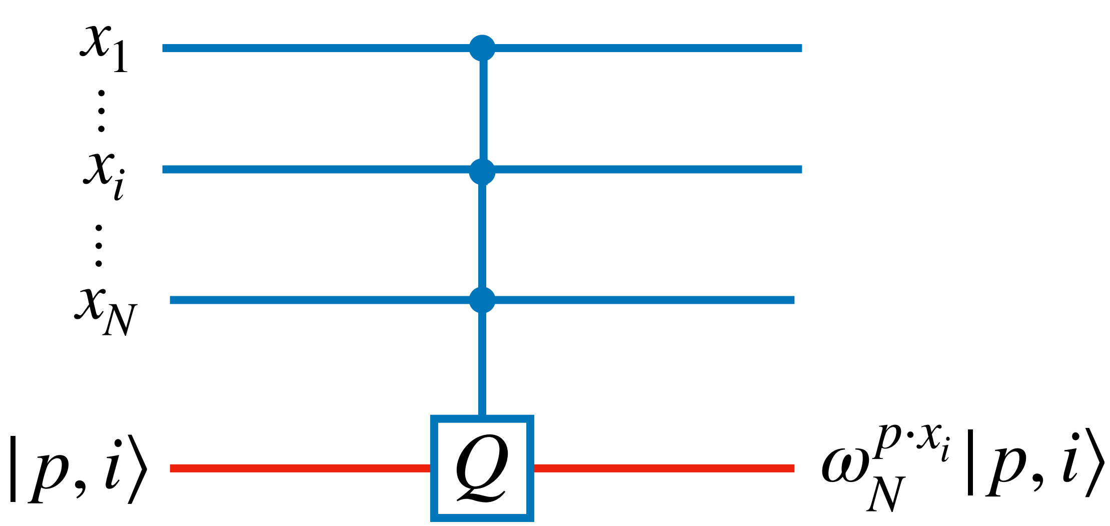


Quantum Recording

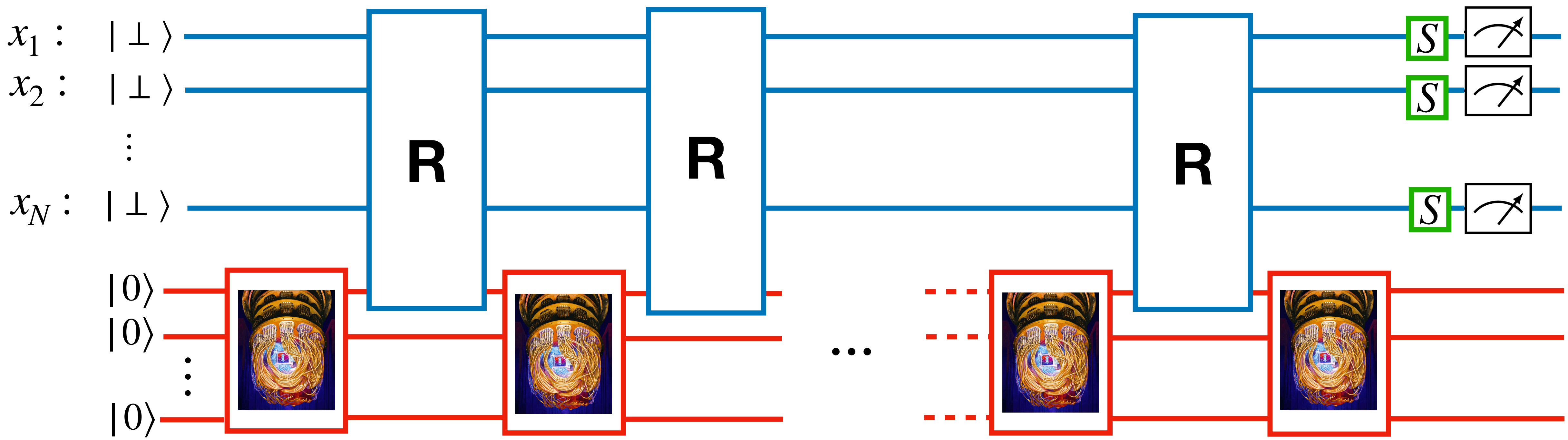


$$|\perp\rangle \; \boxed{S} \; N^{-1/2}\sum_{y\in[N]}|y\rangle$$

Query Operator

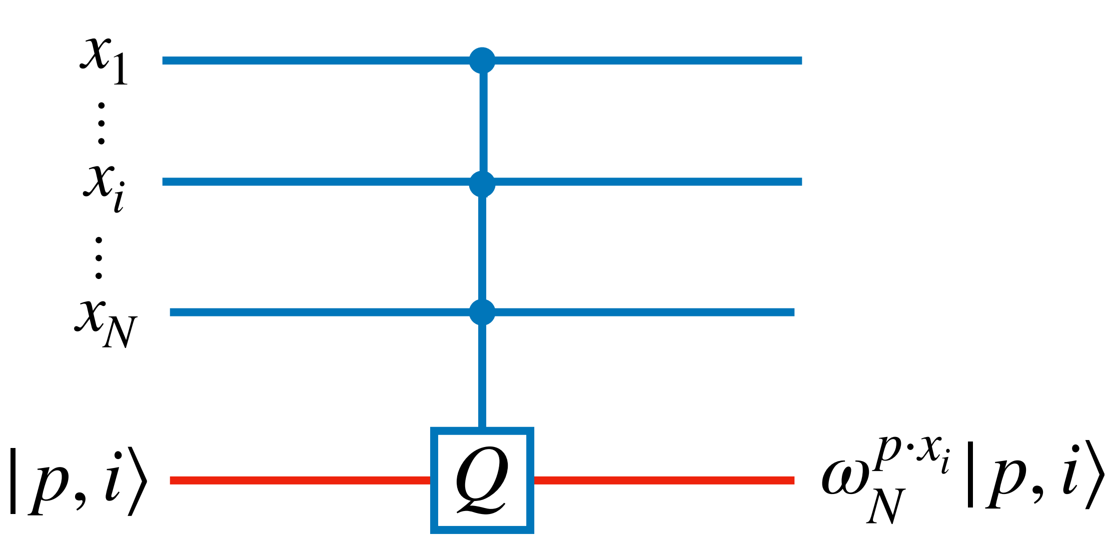


Quantum Recording

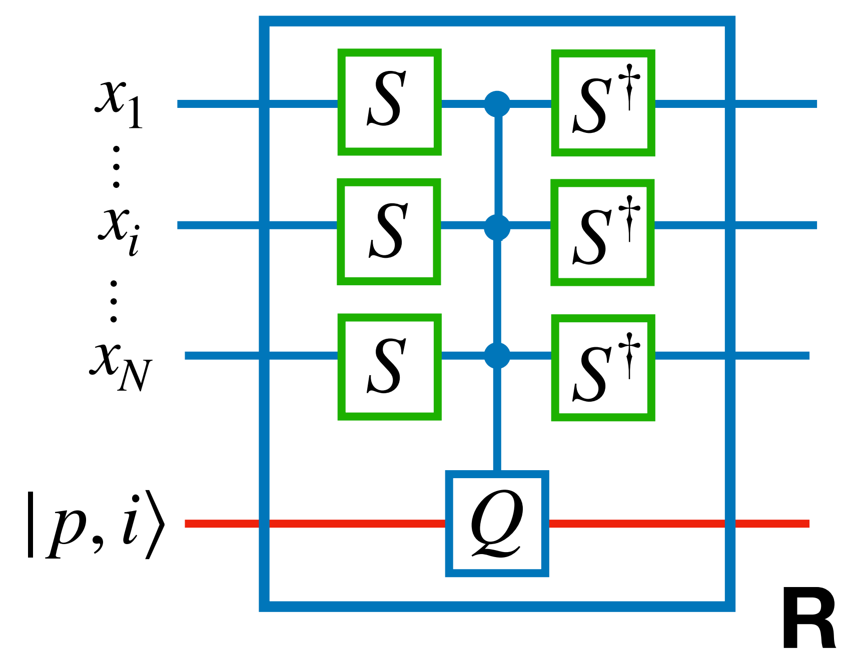


$$|\perp\rangle \xrightarrow{S} N^{-1/2}\sum_{y\in[N]}|y\rangle$$

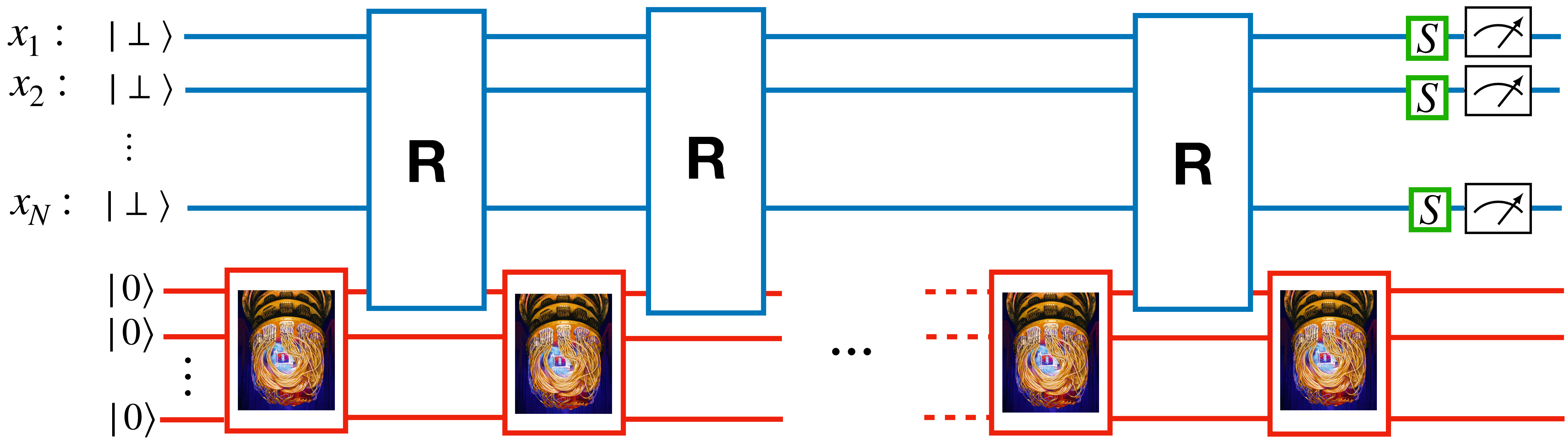
Query Operator



Recording Query Operator



Quantum Recording

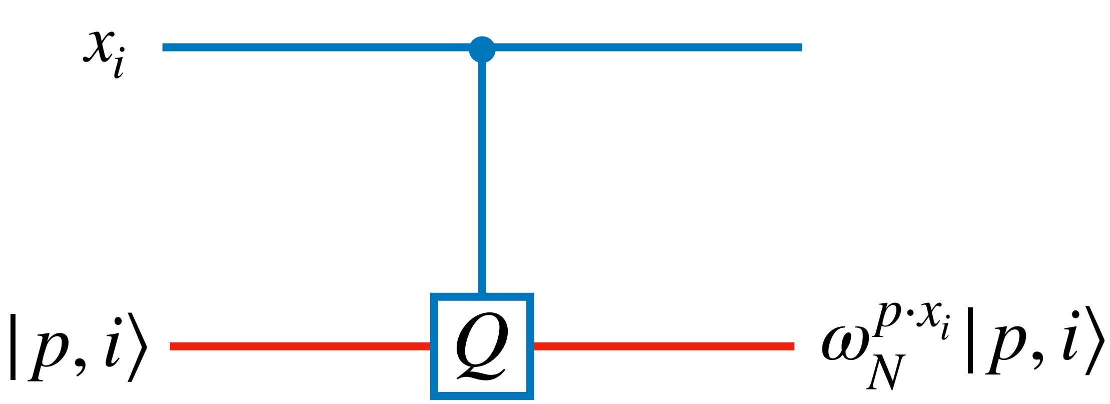


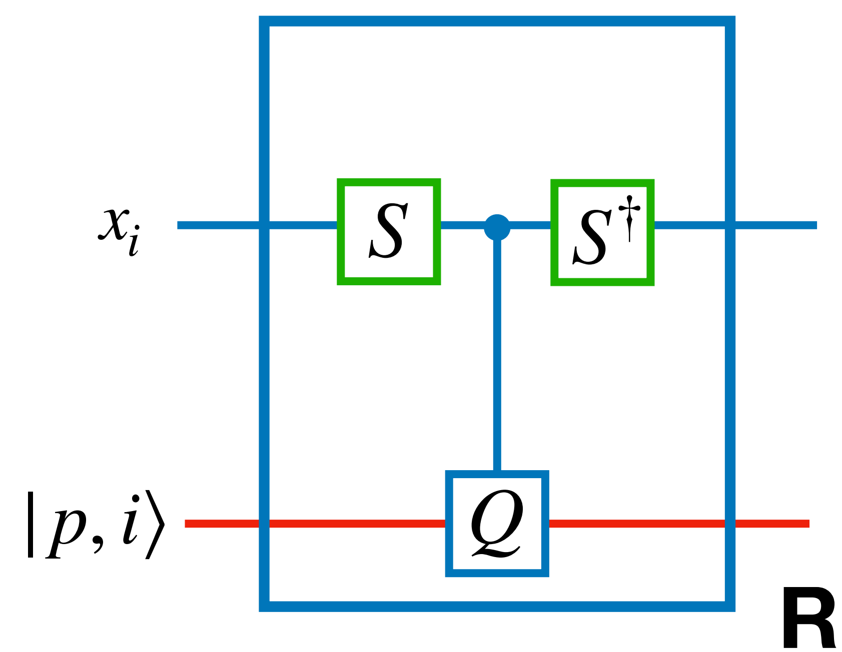
$|\perp\rangle$ — $\boxed{S}$ — $N^{-1/2}\sum_{y\in[N]}|y\rangle$

Query Operator

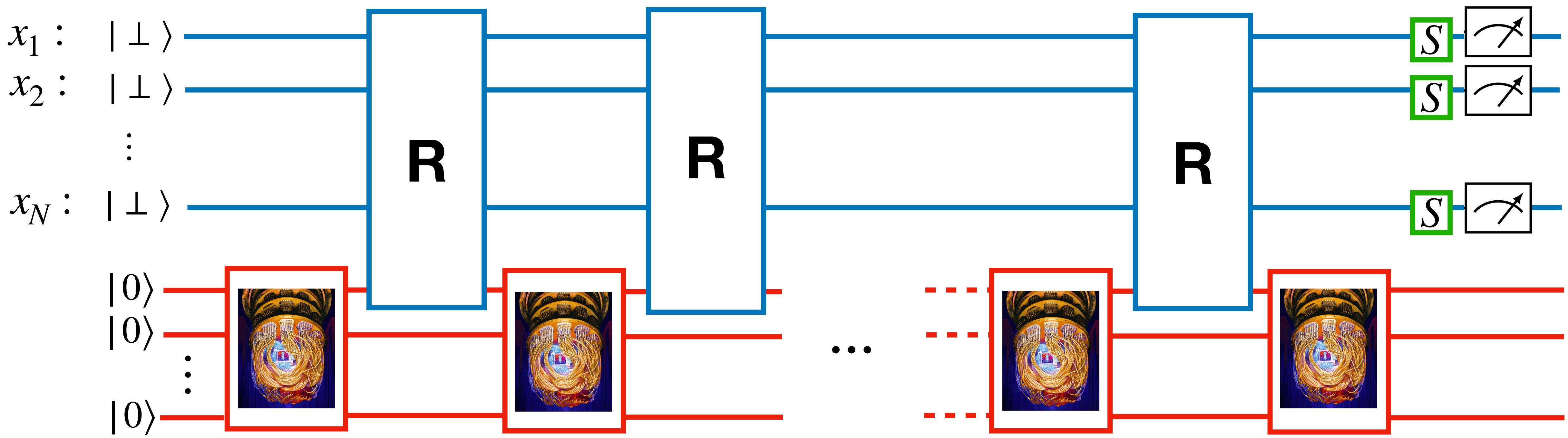
Recording Query Operator

✔ If x_j is not queried it stays unchanged.

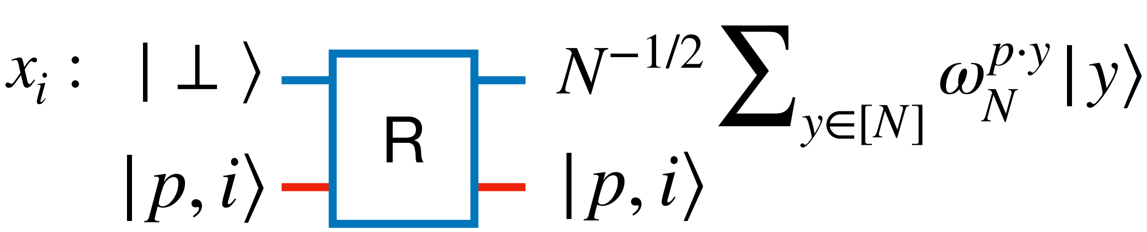




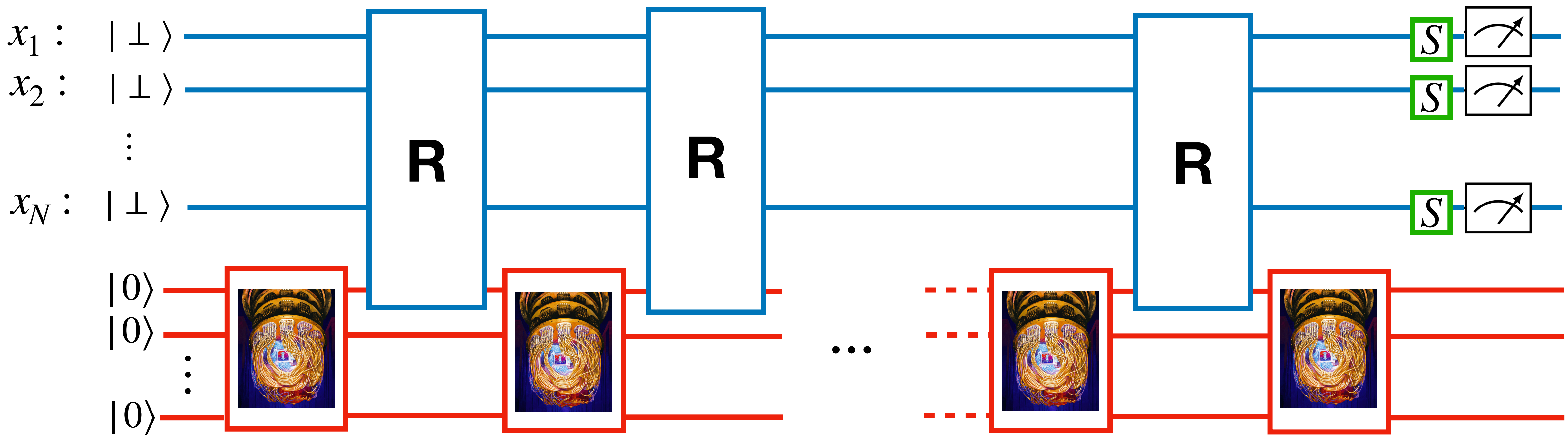
Quantum Recording



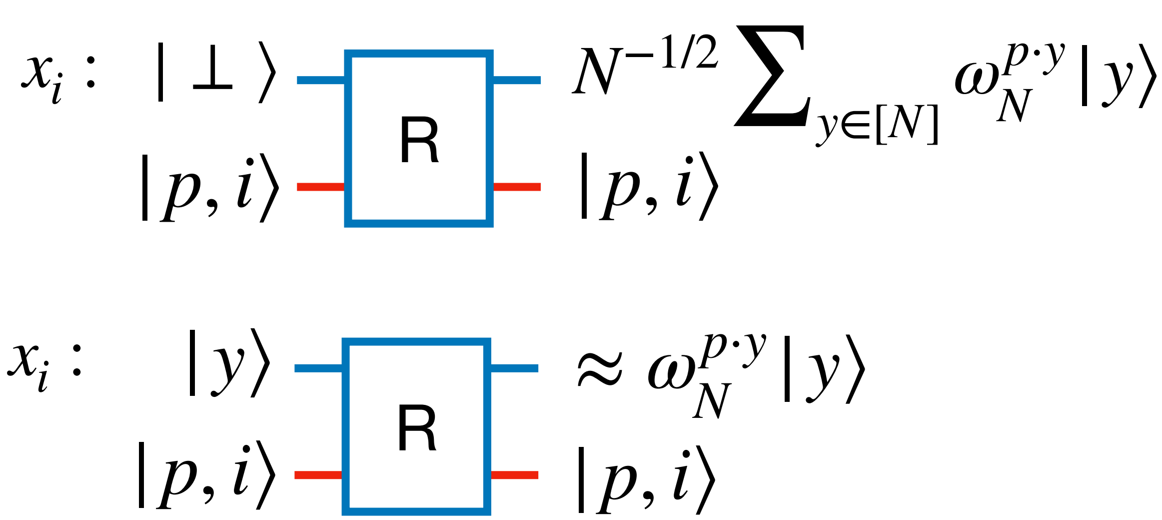
Recording Query Operator



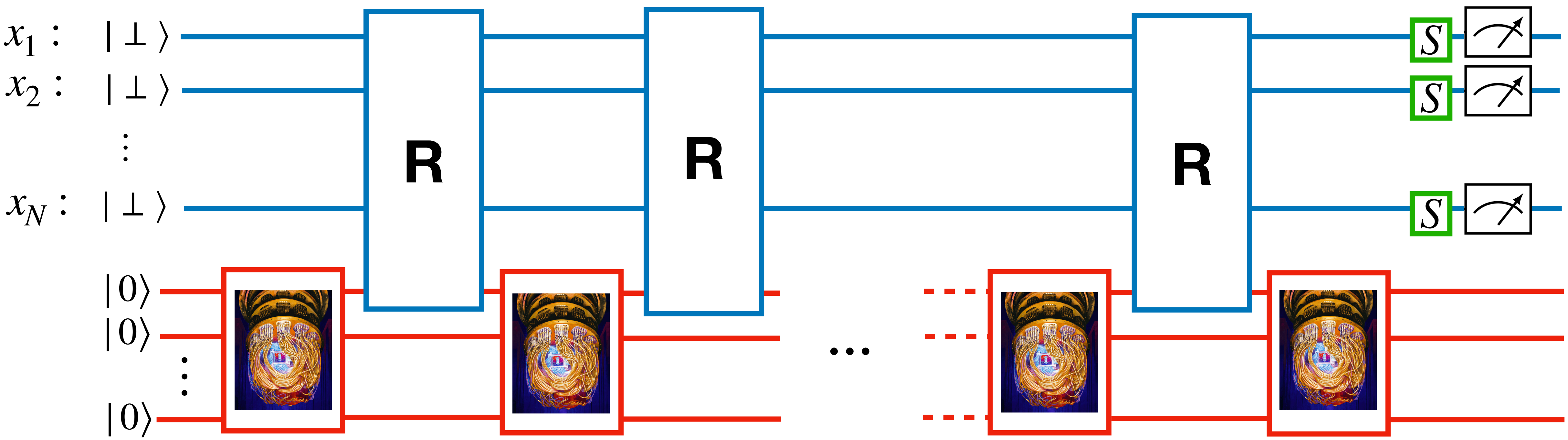
Quantum Recording



Recording Query Operator



Quantum Recording



Recording Query Operator

$$x_i: \quad |\perp\rangle \;\text{—}\boxed{\mathrm{R}}\text{—}\; N^{-1/2}\sum_{y\in[N]}\omega_N^{p\cdot y}|y\rangle$$

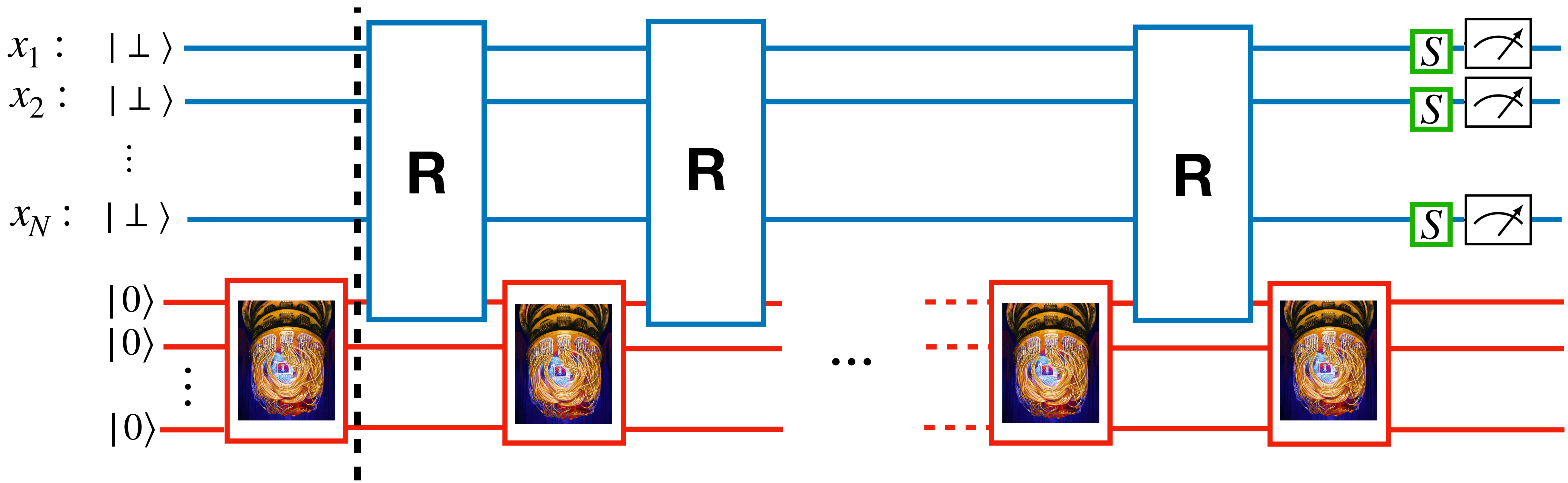
$$|p,i\rangle \;\text{—}\boxed{\mathrm{R}}\text{—}\; |p,i\rangle$$

$$x_i: \quad |y\rangle \;\text{—}\boxed{\mathrm{R}}\text{—}\; \approx \omega_N^{p\cdot y}|y\rangle$$

$$|p,i\rangle \;\text{—}\boxed{\mathrm{R}}\text{—}\; |p,i\rangle$$

$\Delta(t,k)$ = **amplitude** of the basis states containing $\geq$ $\boldsymbol{k}$ (disjoint) collisions after $\boldsymbol{t}$ queries.

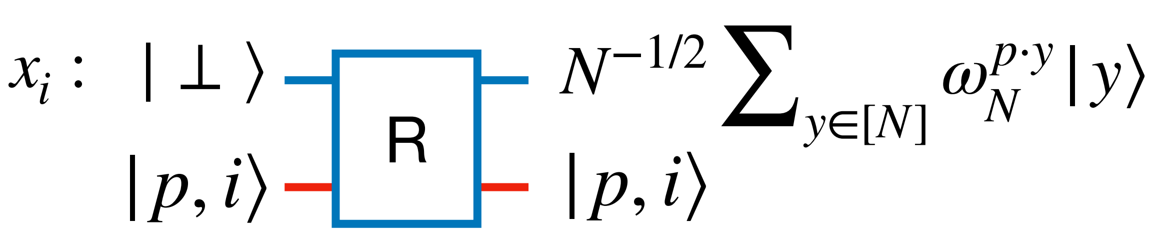
Quantum Recording

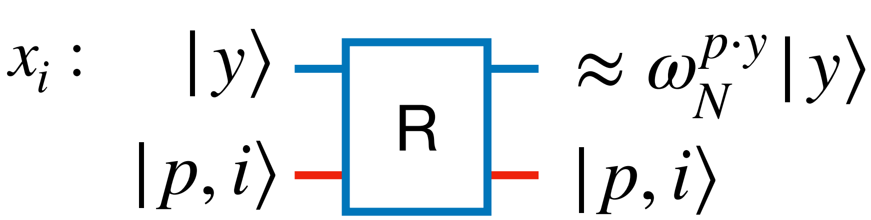


$\Delta(0,0) = 1$

$\Delta(0,k) = 0$ for $k > 0$

Recording Query Operator





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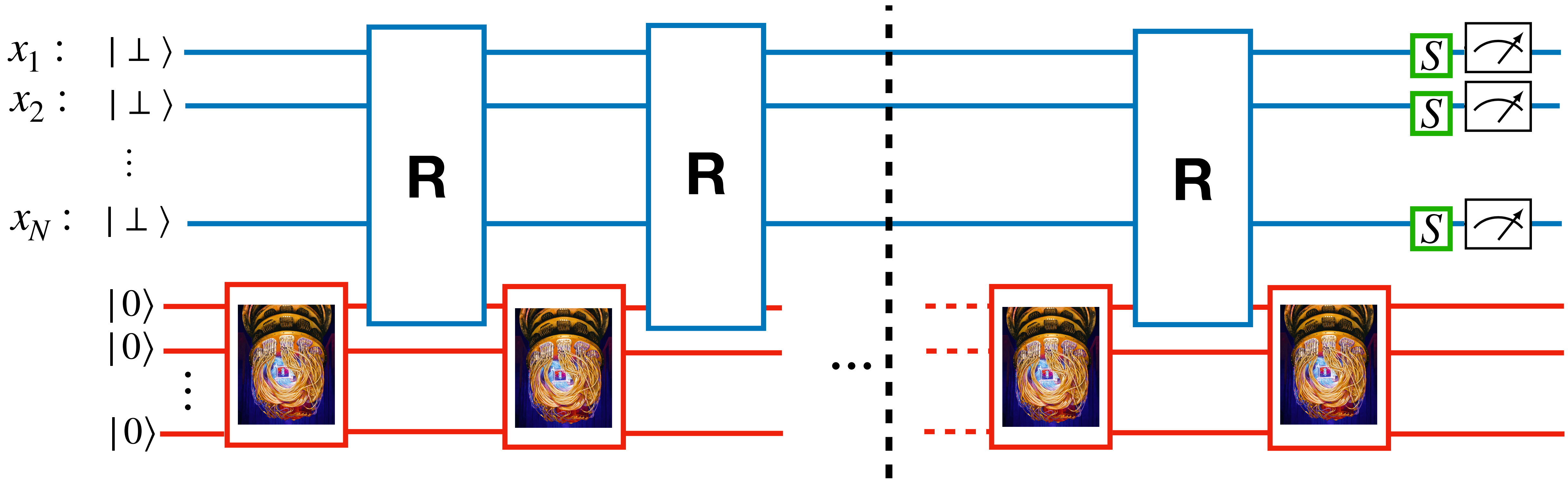
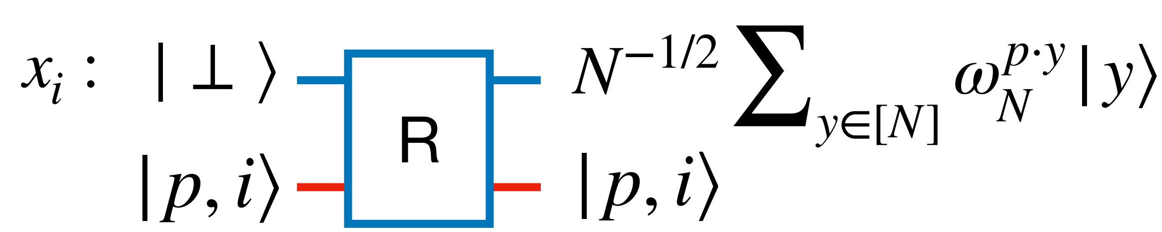
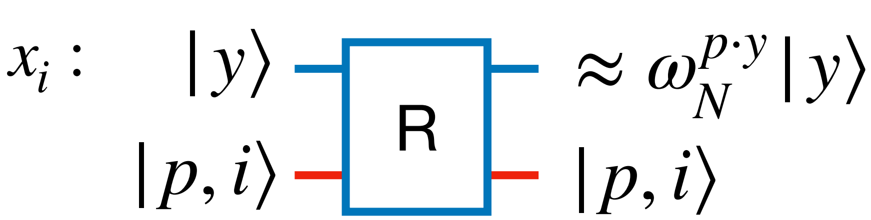
Quantum Recording



$$\Delta(t+1, k+1) \le \Delta(t, k+1) + \Delta(t,k) \cdot O\left(\sqrt{\frac{t}{N}}\right)$$

Recording Query Operator





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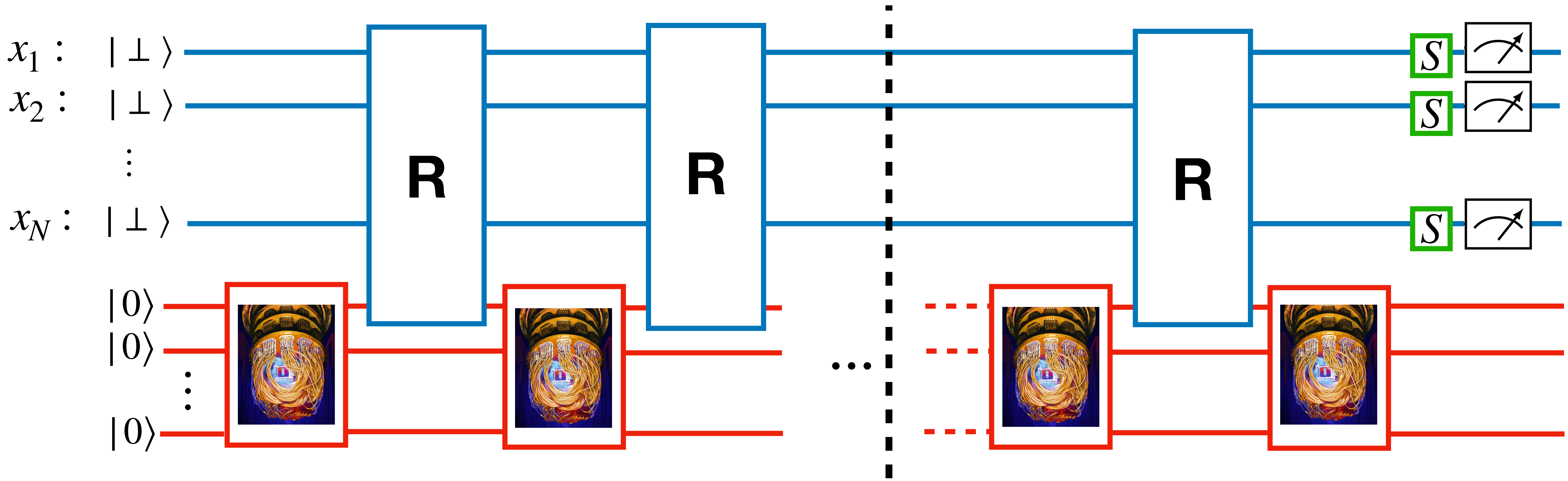


$$\Delta(t+1, k+1) \leq \Delta(t, k+1) + \Delta(t,k) \cdot O\left(\sqrt{\frac{t}{N}}\right)$$

Recording Query Operator

Record

$x_i : \ |\perp\rangle$ —[R]— $N^{-1/2} \sum_{y \in [N]} \omega_N^{p\cdot y} |y\rangle$

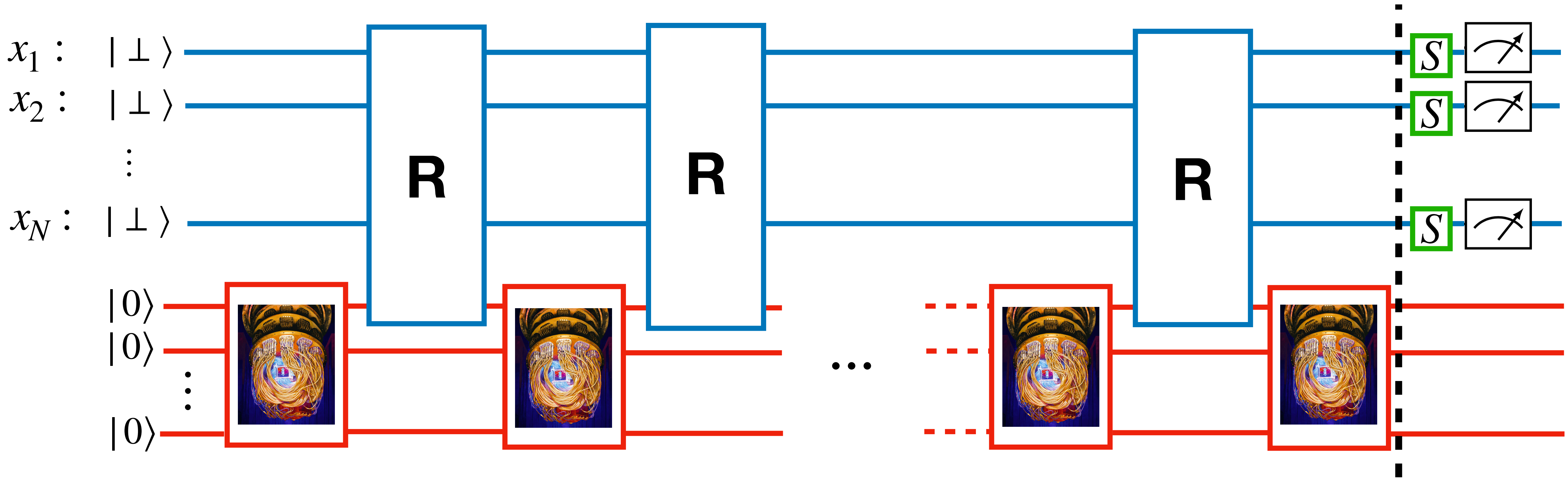
$|p, i\rangle$ —[R]— $|p, i\rangle$

$x_i : \ |y\rangle$ —[R]— $\approx \omega_N^{p \cdot y} |y\rangle$

$|p, i\rangle$ —[R]— $|p, i\rangle$

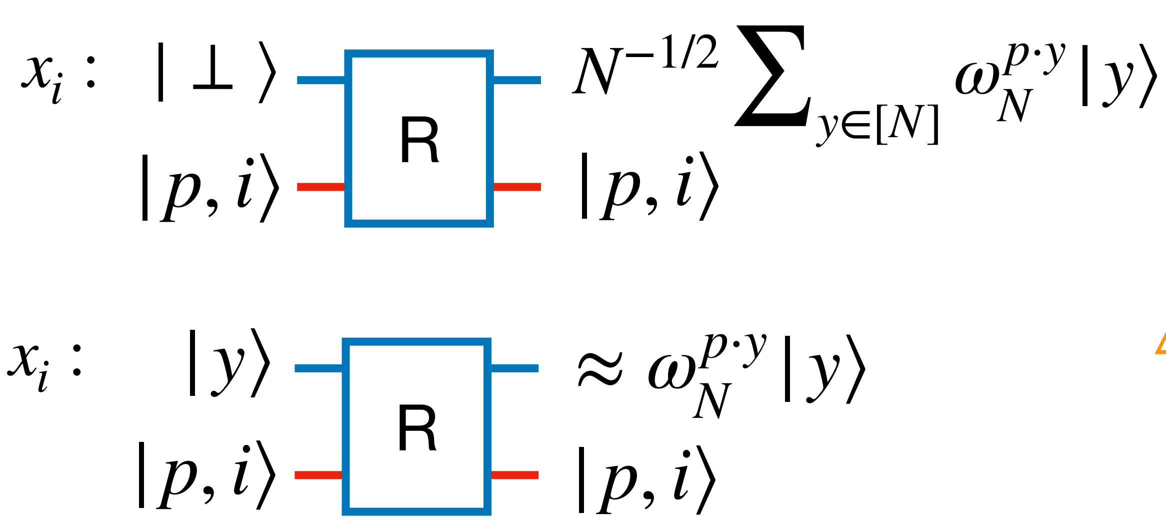
$\Delta(t, k)$ = **amplitude** of the basis states containing $\geq \boldsymbol{k}$ (disjoint) collisions after $\boldsymbol{t}$ queries.

Quantum Recording



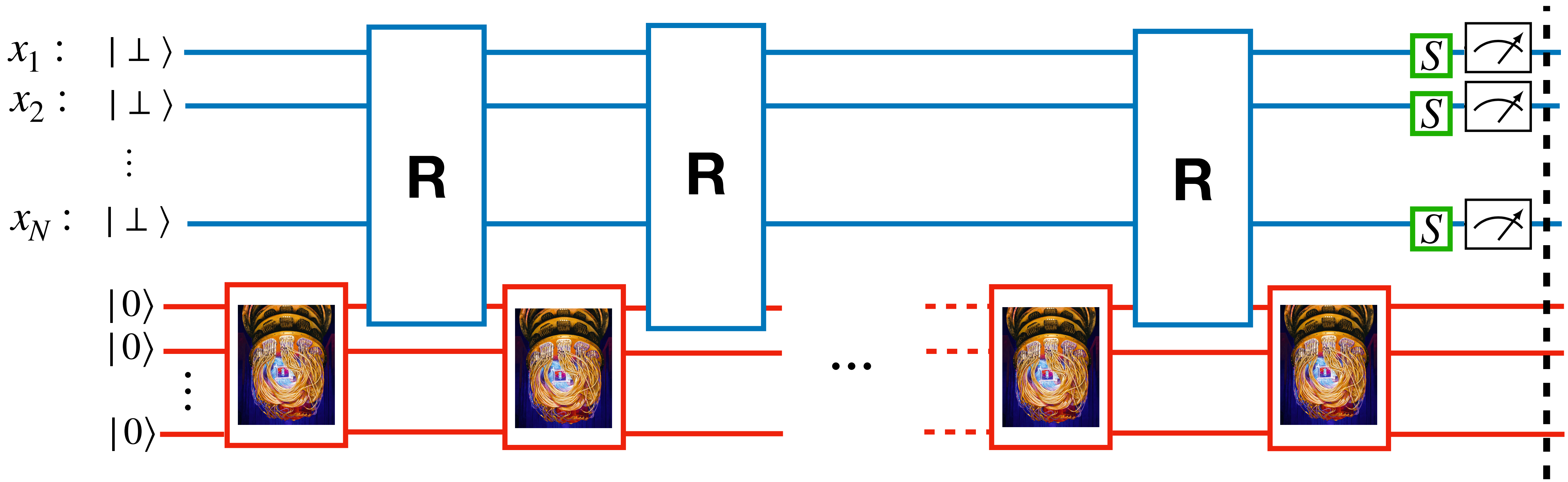
$\Delta(K^{2/3}N^{1/3}, K/2) \leq 2^{-K}$

Recording Query Operator



$\Delta(t, k)$ = **amplitude** of the basis states containing $\geq$ $\boldsymbol{k}$ (disjoint) collisions after $\boldsymbol{t}$ queries.

Quantum Recording



Success
≤
$\Delta(K^{2/3}N^{1/3}, K/2)$
$+\ O(K/N)^{K/2}$

Recording Query Operator

$x_i :\ |\perp\rangle$ — R — $N^{-1/2}\sum_{y\in[N]} \omega_N^{p\cdot y}\,|y\rangle$

$|p,i\rangle$ — R — $|p,i\rangle$

$x_i :\ |y\rangle$ — R — $\approx \omega_N^{p\cdot y}\,|y\rangle$

$|p,i\rangle$ — R — $|p,i\rangle$

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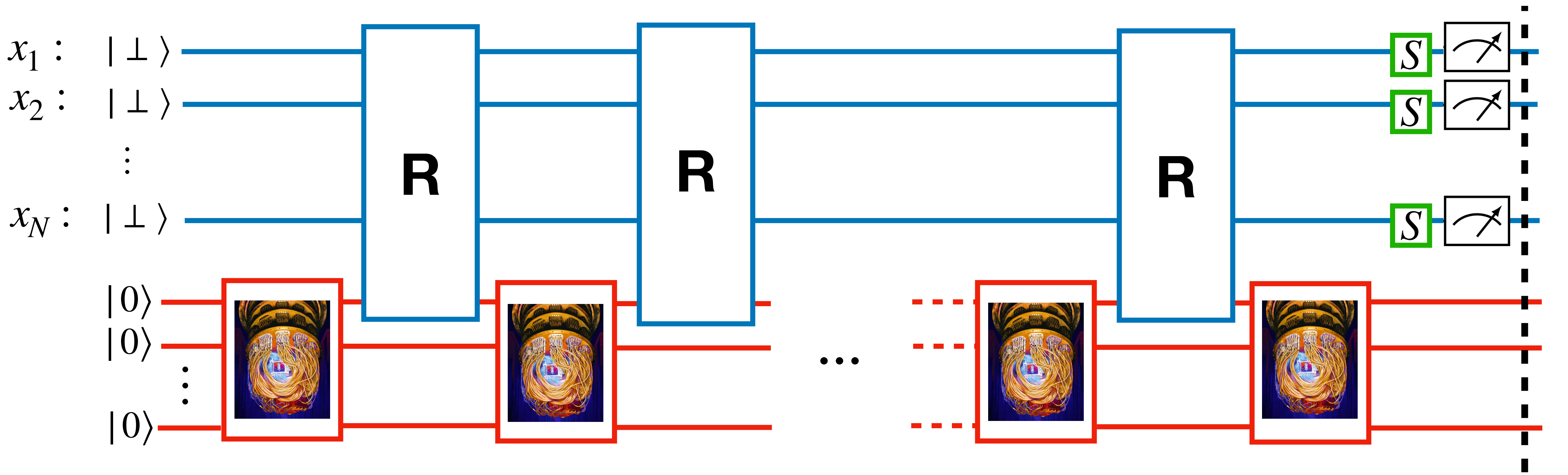


Success
≤
$\Delta(K^{2/3}N^{1/3}, K/2)$
$+ \; O(K/N)^{K/2}$

≈ guess K/2 unrecorded collisions

Recording Query Operator

$x_i : \; |\perp\rangle \xrightarrow{\text{R}} N^{-1/2} \sum_{y \in [N]} \omega_N^{p \cdot y} |y\rangle$

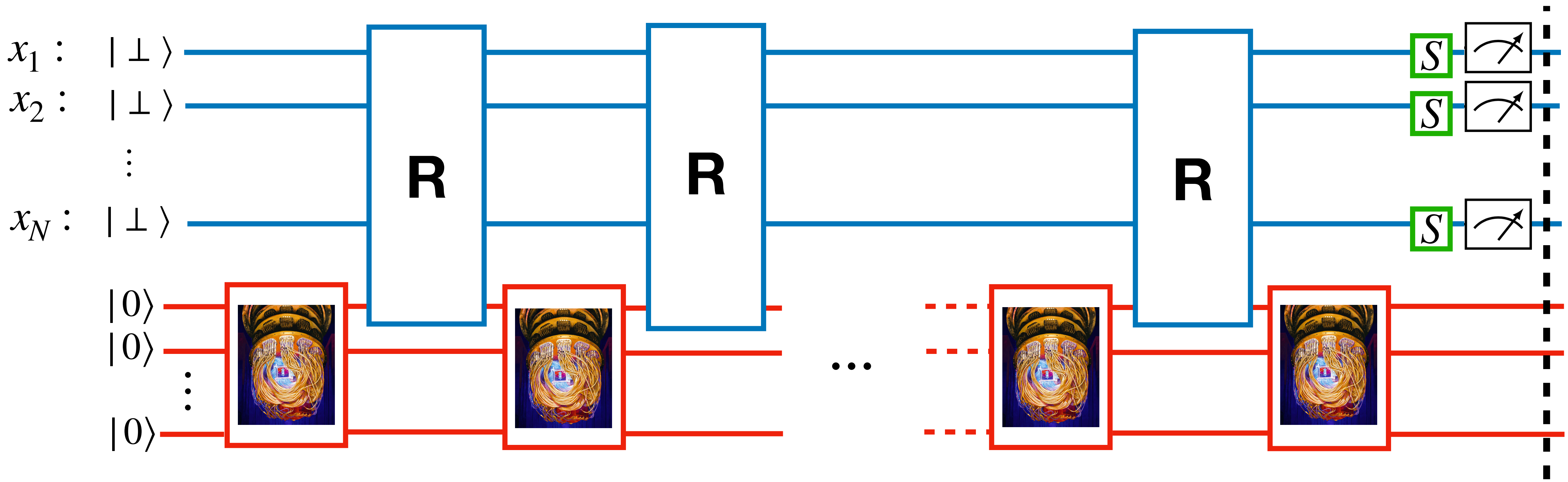
$|p, i\rangle \xrightarrow{\text{R}} |p, i\rangle$

$x_i : \; |y\rangle \xrightarrow{\text{R}} \approx \omega_N^{p \cdot y} |y\rangle$

$|p, i\rangle \xrightarrow{\text{R}} |p, i\rangle$

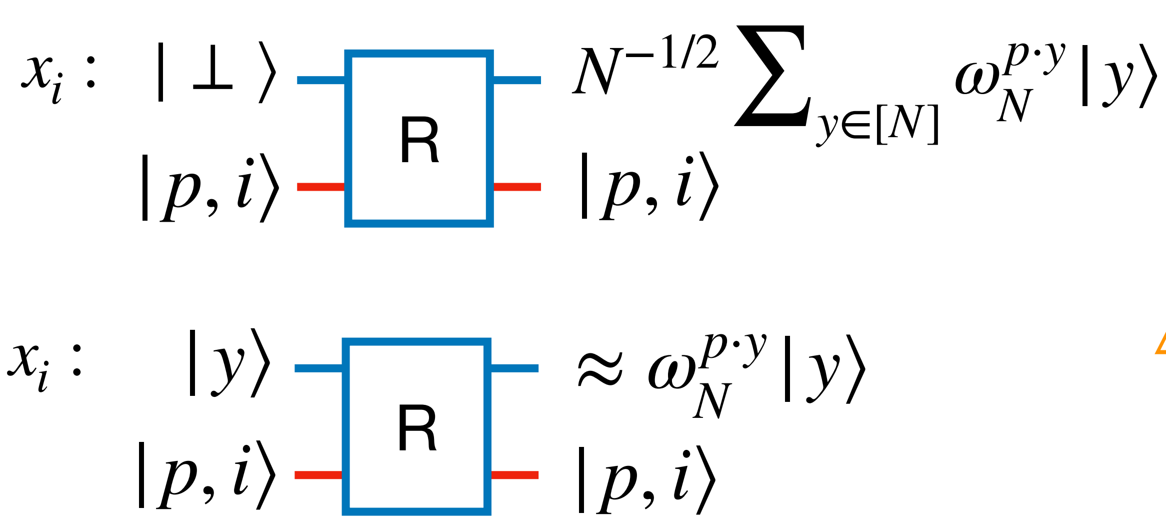
$\Delta(t, k)$ = **amplitude** of the basis states containing $\geq \boldsymbol{k}$ (disjoint) collisions after $\boldsymbol{t}$ queries.

Quantum Recording



Success
≤
$\Delta(K^{2/3}N^{1/3}, K/2)$
$+ \; O(K/N)^{K/2}$
$\lesssim 2^{-\Omega(K)}$

Recording Query Operator



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Conclusion

- Generalization to any product distribution

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→ K-Search: find K ones in x where $x_i = 1$ with probability K/N

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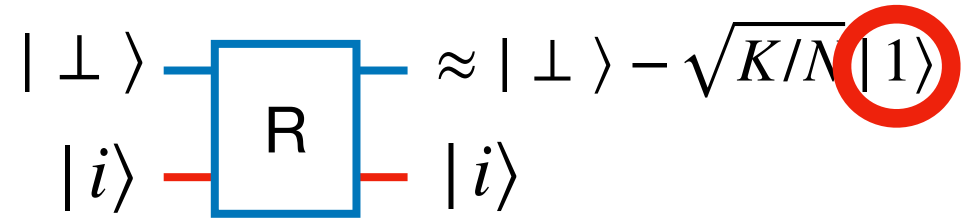
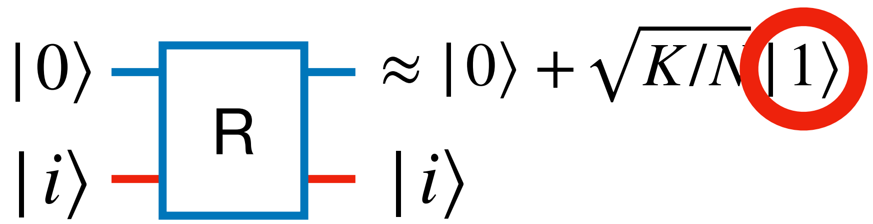
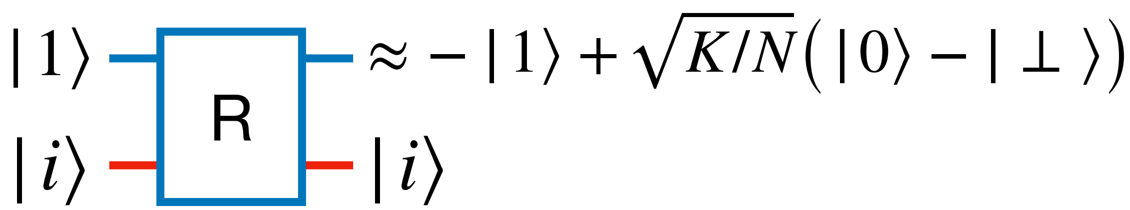


Record a new 1





- **Generalization to any product distribution**

→ K-Search: find K ones in x where $x_i = 1$ with probability K/N

$$\begin{array}{r} |\perp\rangle \\ |i\rangle \end{array} \boxed{\;R\;} \begin{array}{l} \approx |\perp\rangle - \sqrt{K/N}|1\rangle \\ |i\rangle \end{array}$$

Record a new 1

$$\begin{array}{r} |0\rangle \\ |i\rangle \end{array} \boxed{\;R\;} \begin{array}{l} \approx |0\rangle + \sqrt{K/N}|1\rangle \\ |i\rangle \end{array}$$

$$\begin{array}{r} |1\rangle \\ |i\rangle \end{array} \boxed{\;R\;} \begin{array}{l} \approx -|1\rangle + \sqrt{K/N}\left(|0\rangle - |\perp\rangle\right) \\ |i\rangle \end{array}$$

Solving the K-Search problem with success probability at least $2^{-O(K)}$ requires time $T \geq \Omega\left(\sqrt{NK}\right)$.

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$$|i\rangle \;\text{—}\boxed{R}\text{—}\; |i\rangle$$

Record a new 1

$$|0\rangle \;\text{—}\boxed{R}\text{—}\; \approx |0\rangle + \sqrt{K/N}|1\rangle$$
$$|i\rangle \;\text{—}\boxed{R}\text{—}\; |i\rangle$$

$$|1\rangle \;\text{—}\boxed{R}\text{—}\; \approx -|1\rangle + \sqrt{K/N}\big(|0\rangle - |\perp\rangle\big)$$
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Solving the K-Search problem with success probability at least $2^{-O(K)}$ requires time $T \geq \Omega(\sqrt{NK})$.

+ Simpler proof than previous work [Klauck et al.'07, Ambainis'10, …]
+ Implies several quantum time-space tradeoffs (e.g. for Sorting).

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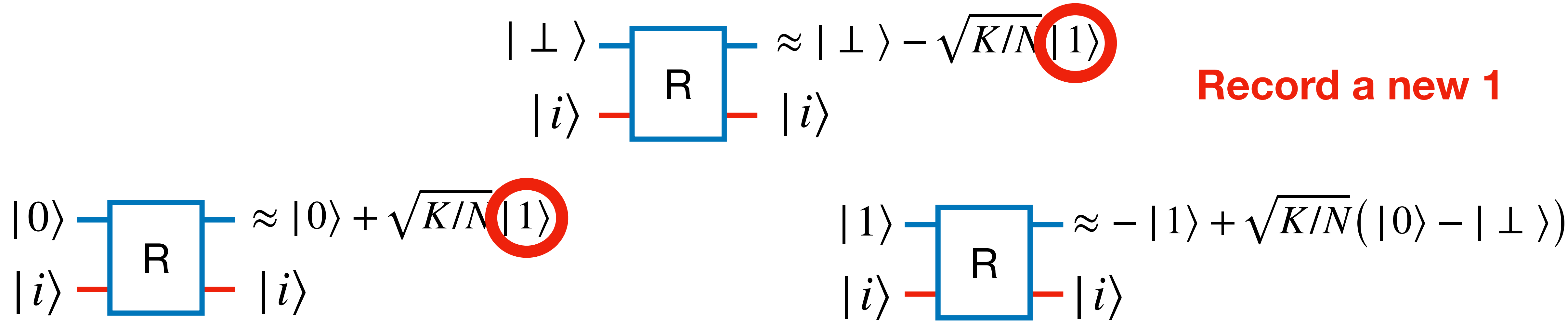

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- Generalization to any product distribution

→ K-Search: find K ones in x where $x_i = 1$ with probability K/N

$$|\perp\rangle|i\rangle \xrightarrow{R} \approx |\perp\rangle - \sqrt{K/N}|1\rangle, \quad |i\rangle$$

Record a new 1

$$|0\rangle|i\rangle \xrightarrow{R} \approx |0\rangle + \sqrt{K/N}|1\rangle, \quad |i\rangle$$

$$|1\rangle|i\rangle \xrightarrow{R} \approx -|1\rangle + \sqrt{K/N}\big(|0\rangle - |\perp\rangle\big), \quad |i\rangle$$

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- Conjecture: $T^2S \geq \Omega(K^2N)$ for finding K collisions.