# Quantum Time-Space Tradeoffs by Recording Queries

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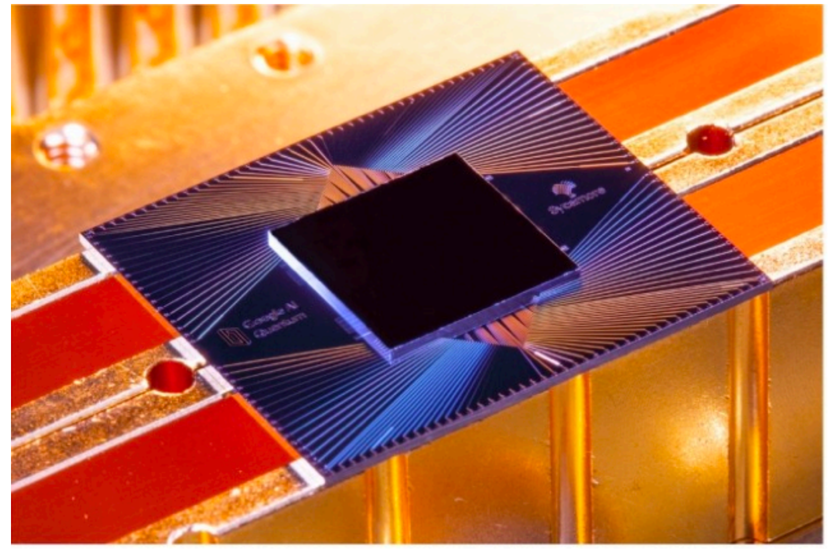
arXiv: 2002.08944

## Google Sycamore's calculation



**Time** ≈ 5 minutes
**Memory** ≈ 53 qubits
+ few megabytes

## Simulation by Schrödinger-Feynman algorithm


**Time** ≈ 10,000 years
**Memory** ≈ 1 Petabyte

## Simulation by Schrödinger algorithm


**Time** ≈ 2.5 days
**Memory** ≈ 250 Petabytes

1. Time and Space in the Query Model

2. The Collision Pairs Finding Problem

3. Lower Bounds by Recording Queries

1

# Time and Space in the Query Model

# Classical Query Model

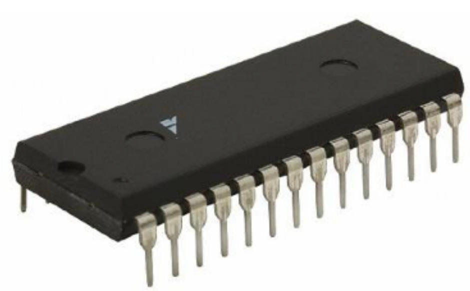


*Input*

*Queries*

*Computer*

$x = (x_1, \ldots, x_N)$



Read Only Memory

$i$

$x_i$

Read-Write Memory

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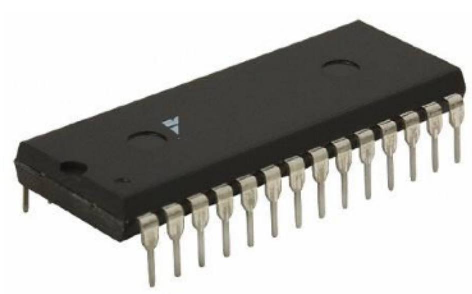

Read Only Memory

$\xleftarrow{\; i \;}$

$\xrightarrow[x_i]{}$

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**Time T** = number of <u>queries</u> to the input

**Space S** = number of <u>bits</u> in the computer's memory

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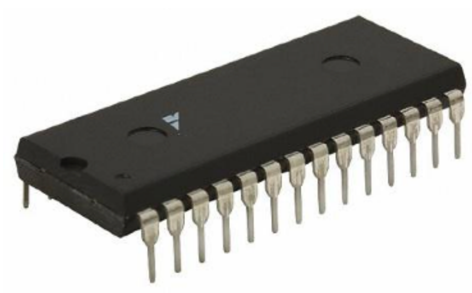

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**Time T** = number of <u>queries</u> to the input

**Space S** = number of <u>bits</u> in the computer's memory

⟶ The number of queries is a lower bound on the actual computation time.

⟶ If S = ∞ then T ≤ N is sufficient (load the entire input in the computer's memory).

⟶ We are interested in the case "T or S << N".

# Classical Query Model



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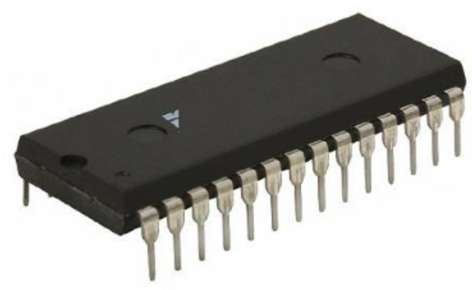

Read Only Memory

$i$ ←

→ $x_i$

Read-Write Memory

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**Time-Space Tradeoffs:**

*[Beame'91]* Sorting N numbers requires time T and space S such that **$TS \geq \Omega(N^2)$**.

# Classical Query Model



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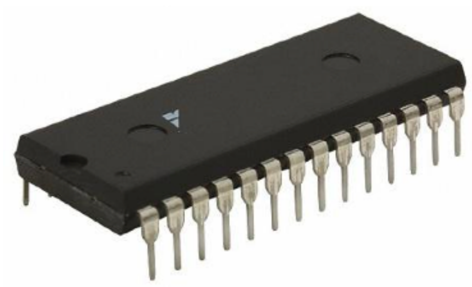

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## Time-Space Tradeoffs:

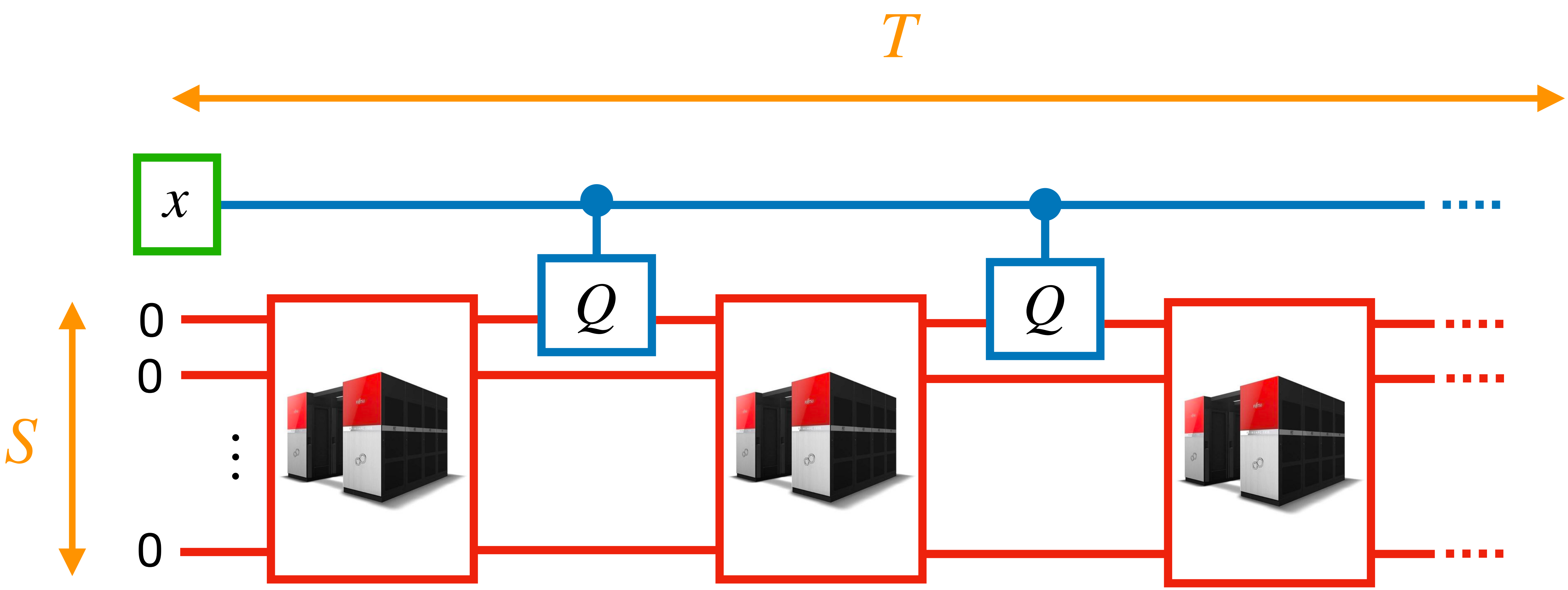
*[Beame'91]* Sorting N numbers requires time T and space S such that **TS ≥ Ω(N²)**.

*[Klauck et al.'07]* Boolean Multiplication of two NxN matrices requires **TS ≥ Ω(N³)**.

…

# Classical Query Model

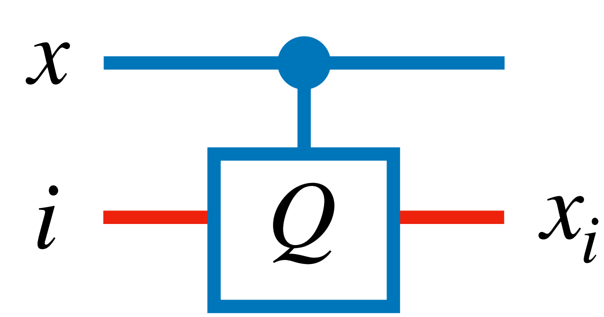





- Initially, the memory is filled with S zeros.

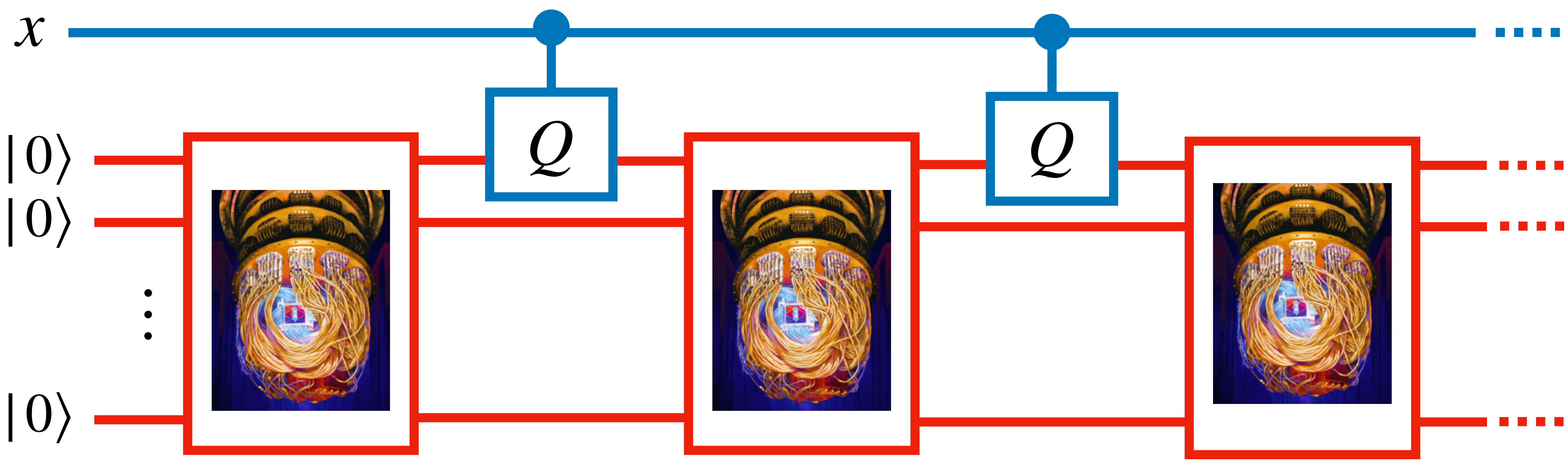
- The computation alternates between T queries and T memory updates.

- The “Query Operator” $Q$ is:



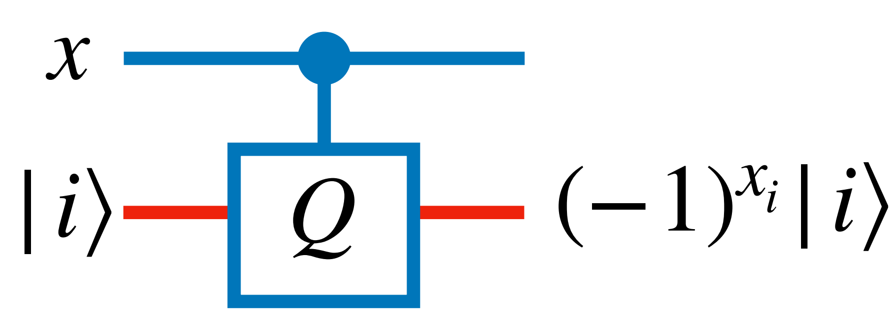
# Quantum Query Model





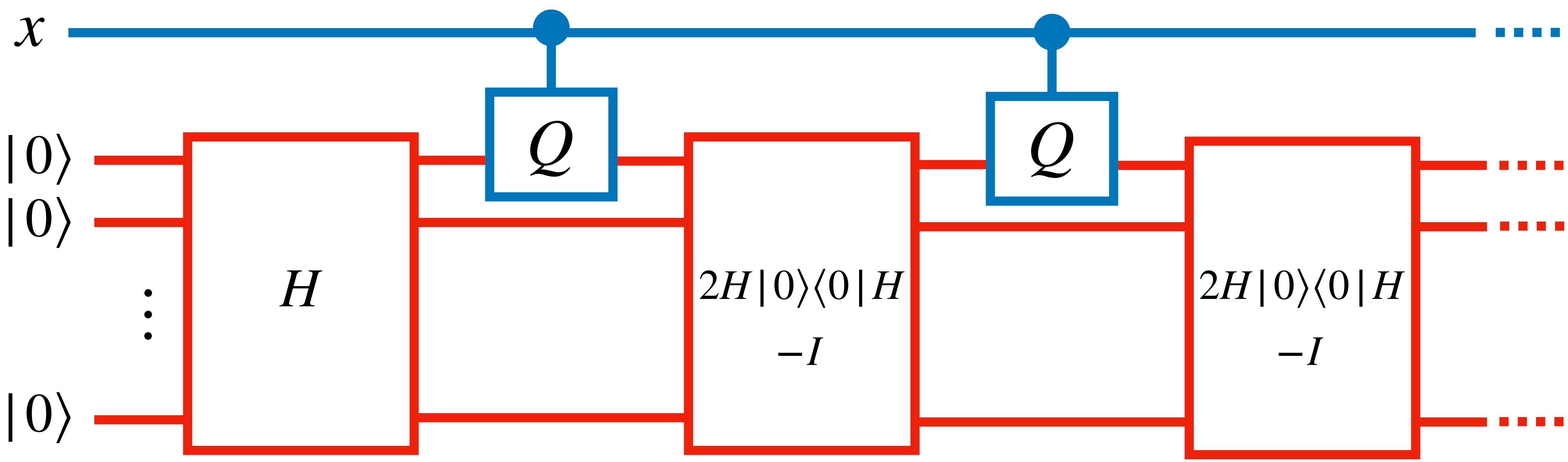

- The memory is made of **S qubits**, initially set to $|0\rangle$.

- The quantum "Query Operator" $Q$ is:
  (when $x_i \in \{0,1\}$)




- The computation alternates between T **quantum queries** and T **unitary updates/ measurements** of the memory.

# Quantum Query Model




**Example:** Grover's Search

$$\begin{cases} T = O\left(\sqrt{N}\right) \\ S = \log(N) \end{cases}$$

# Quantum Time-Space Tradeoffs



**Our focus in this talk:** quantum time-space tradeoff **lower bounds**.

**Very few existing results:**

*[Klauck et al.'07]* Sorting N numbers requires $\mathbf{T^2S \geq \Omega(N^3)}$.

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*[Klauck et al.'07]* Sorting N numbers requires $\mathbf{T^2S \geq \Omega(N^3)}$.

*[Klauck et al.'07]* Boolean Matrix-Matrix Multiplication requires $\mathbf{T^2S \geq \Omega(N^5)}$.

*[Klauck et al.'07]* Boolean Matrix-Vector Multiplication requires $\mathbf{T^2S \geq \Omega(N^3)}$.

*[Ambainis et al.'09]* Evaluating $Ax \geq (t,\ldots,t)$ requires $\mathbf{T^2S \geq \Omega(tN^3)}$ when $S < N/t$.

$\mathbf{TS \geq \Omega(N^2)}$ when $S > N/t$.

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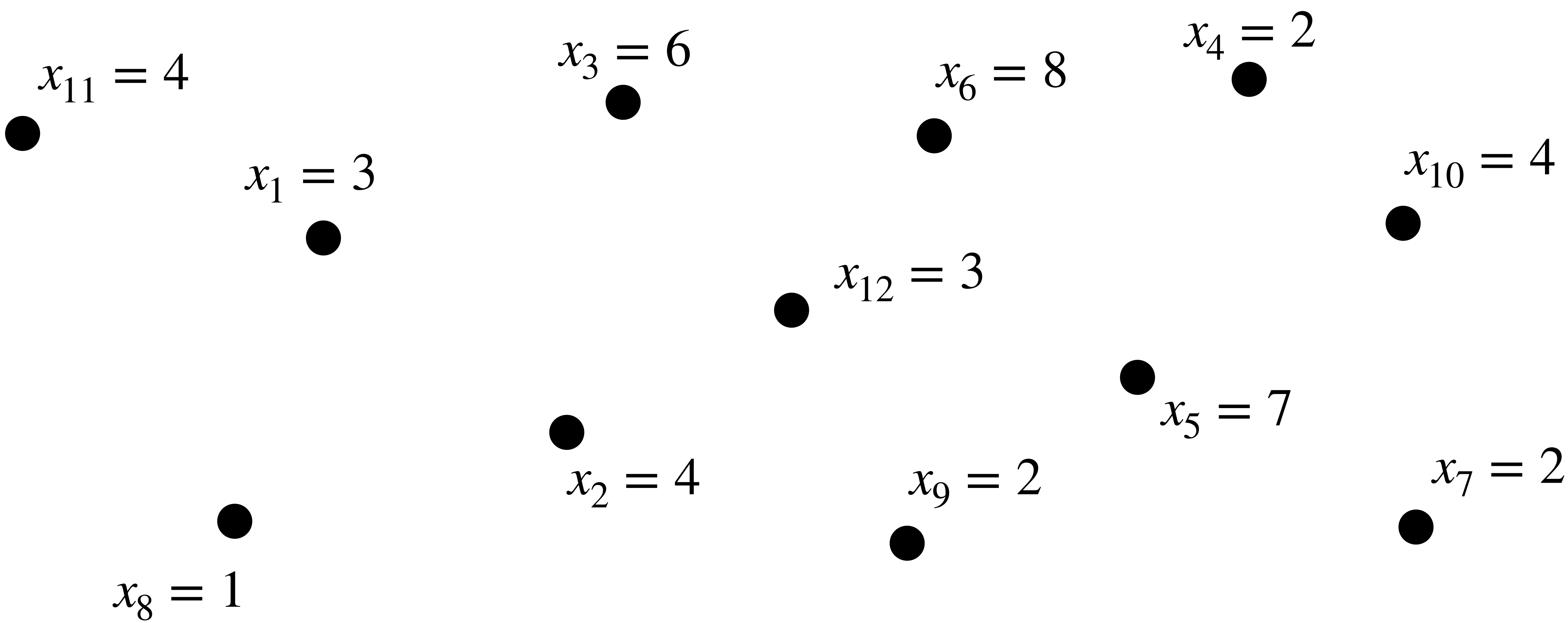
**Our contribution:** a new tradeoff for the **Collision Pairs Finding** problem.

2

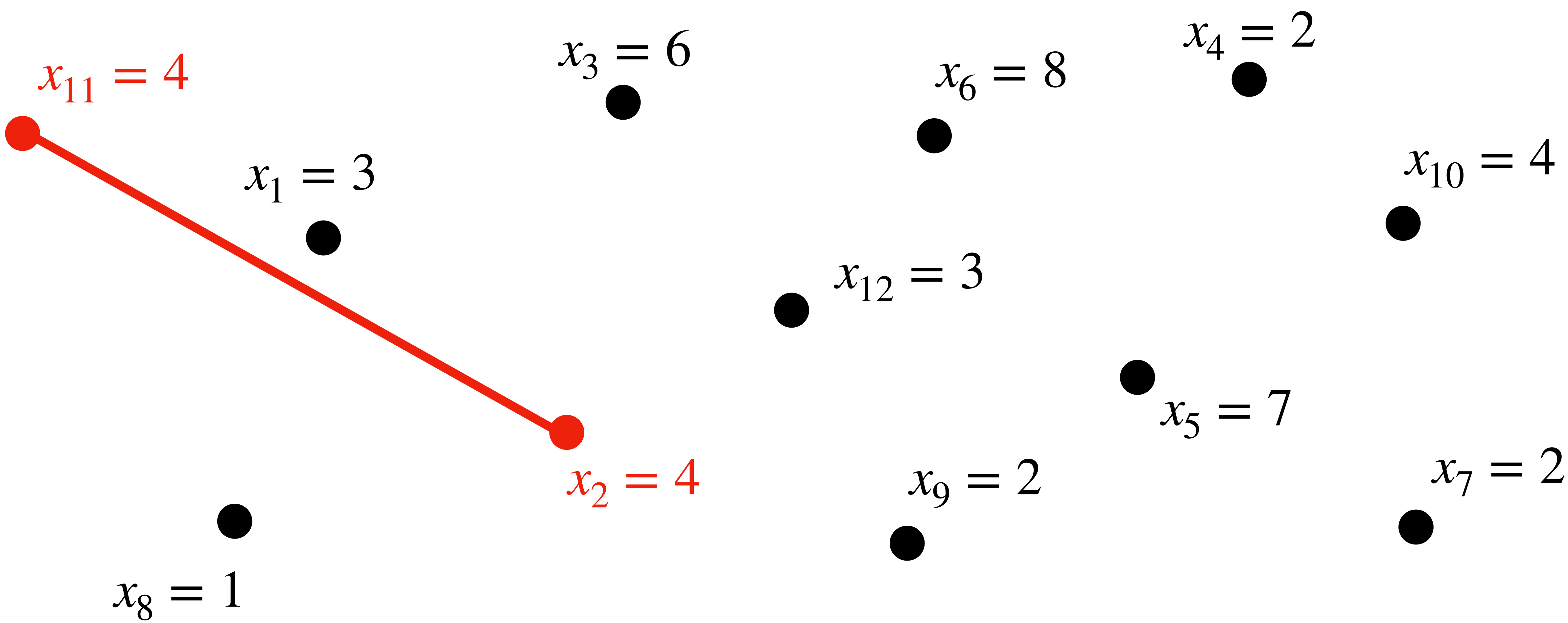
# The Collision Pairs Finding Problem

# Collision Pairs




**Collision pair: $x_i = x_j$**

# Collision Pairs

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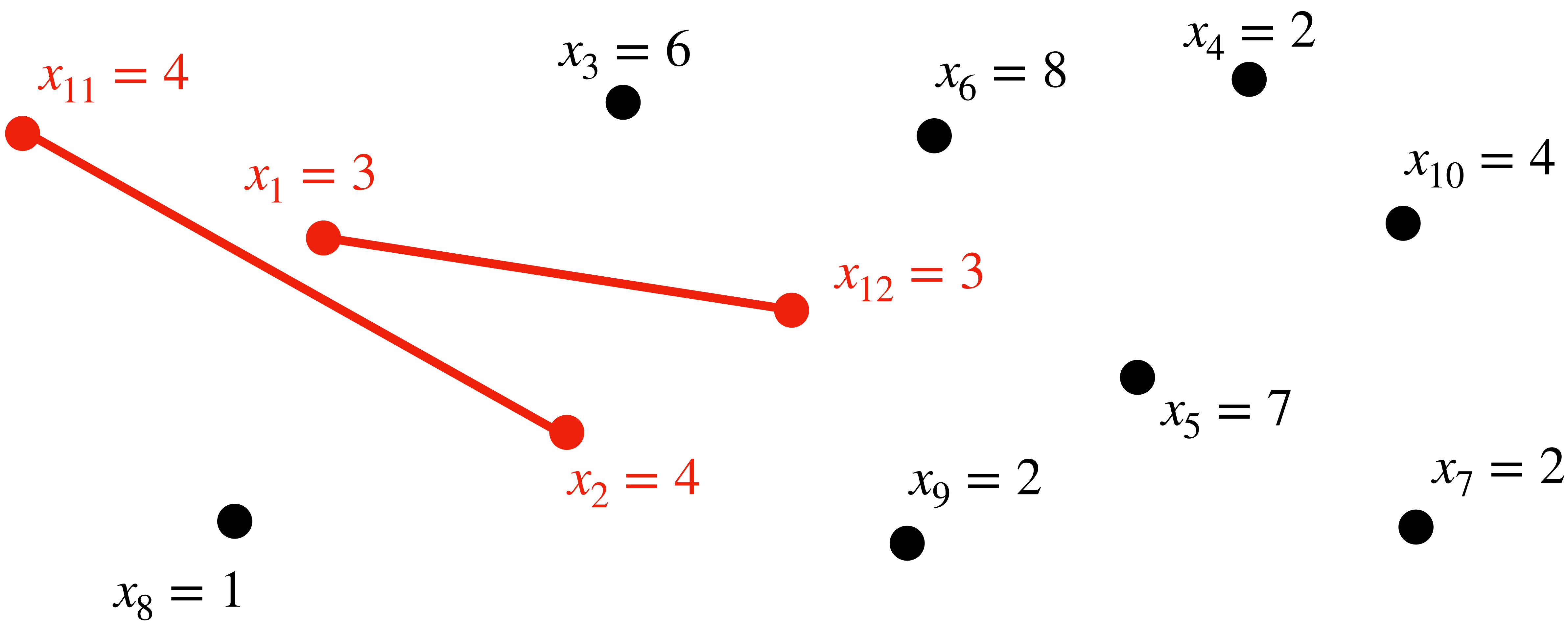


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## K-Collision Pairs

Find K collision pairs in a **random** input $x_1, \ldots, x_N$ ~ [N].

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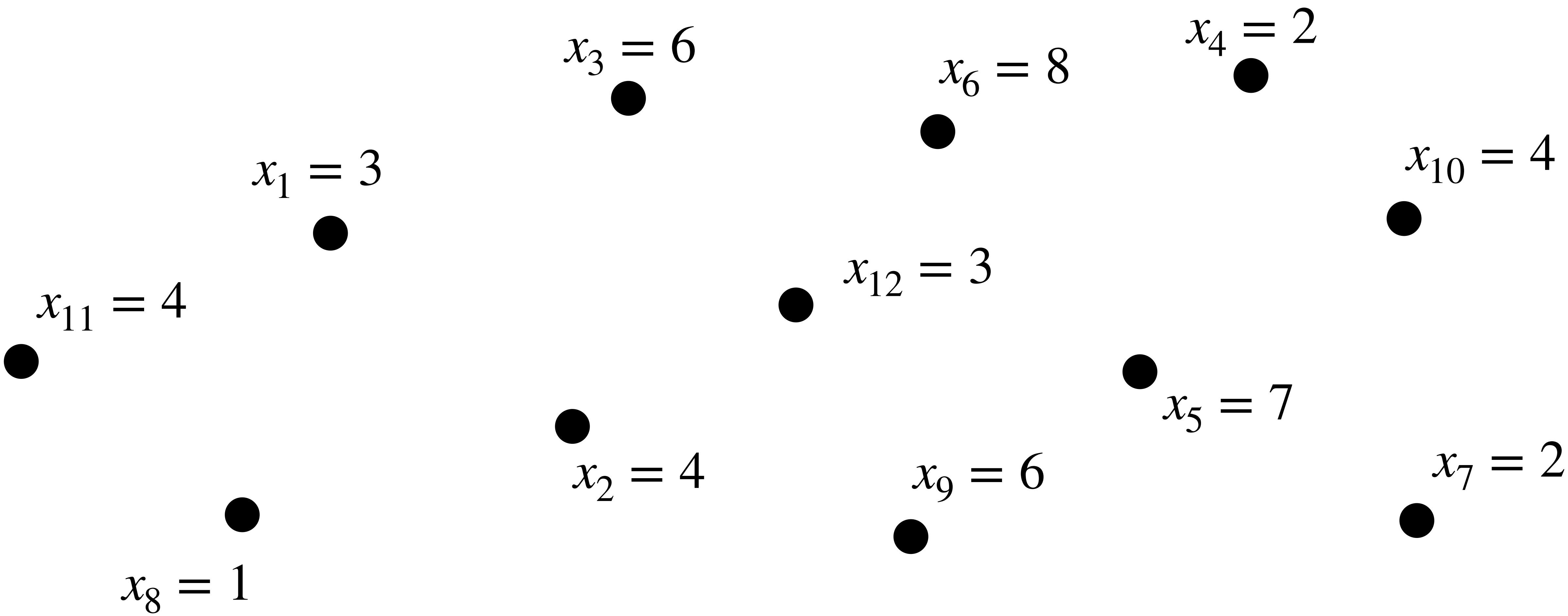
→ **A random input contains ~ Θ(N) collision pairs with high probability.**

→ **Finding collisions is an important problem in cryptanalysis:**

- preimage attacks on hash functions
- meet-in-the-middle attacks ← **requires to find many collisions**
- computing discrete logarithms
- …

# Birthday attack (K = 1)





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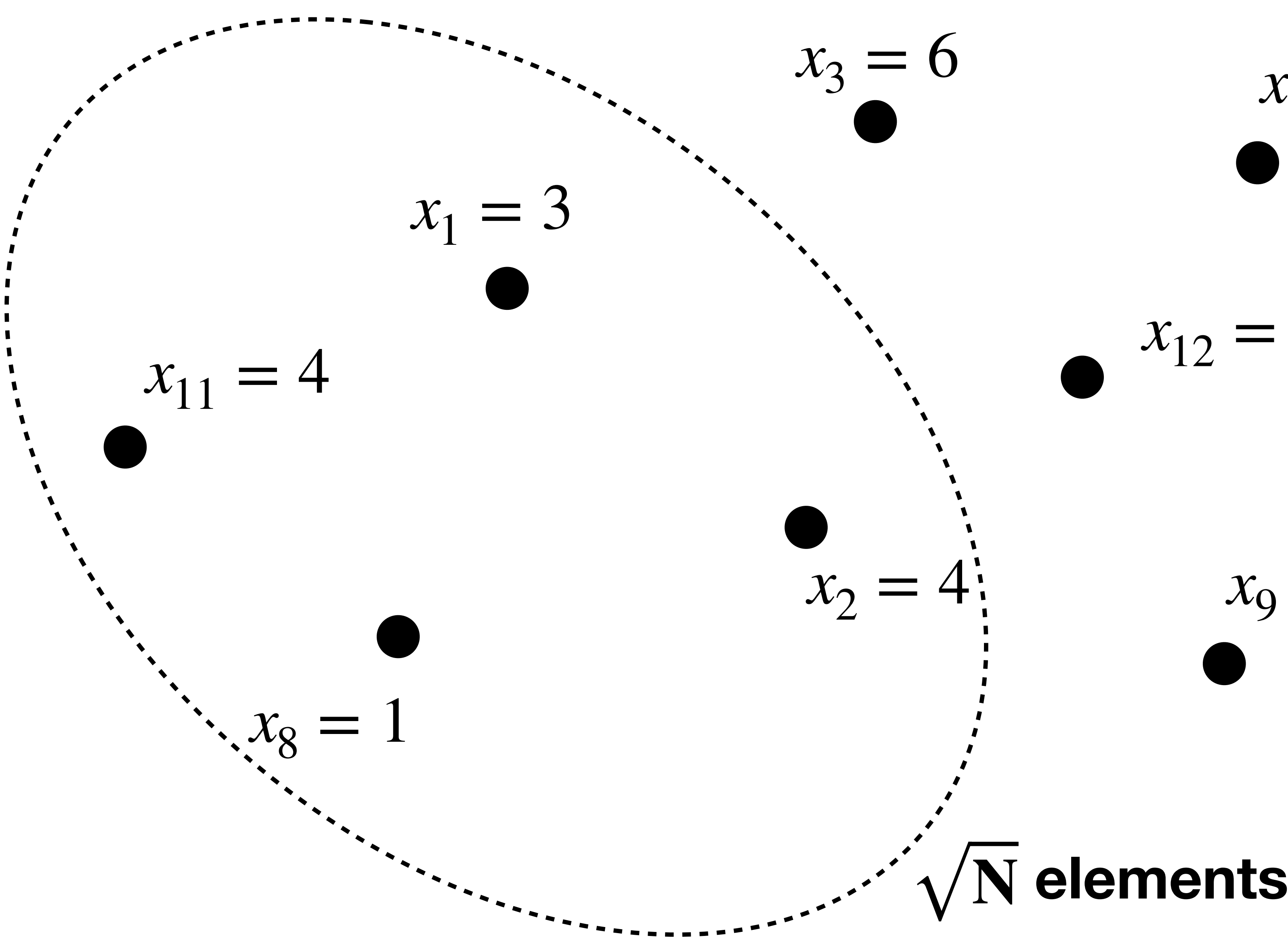


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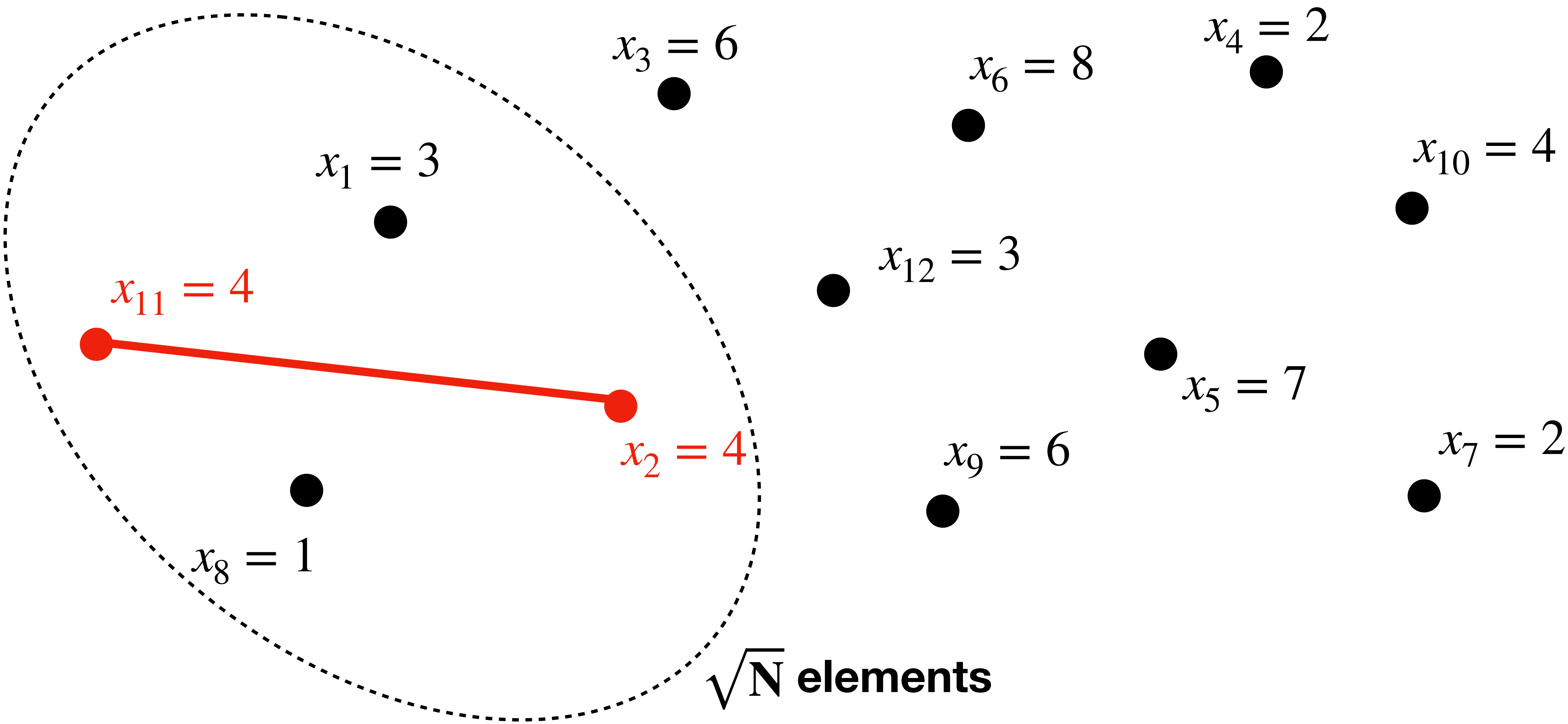


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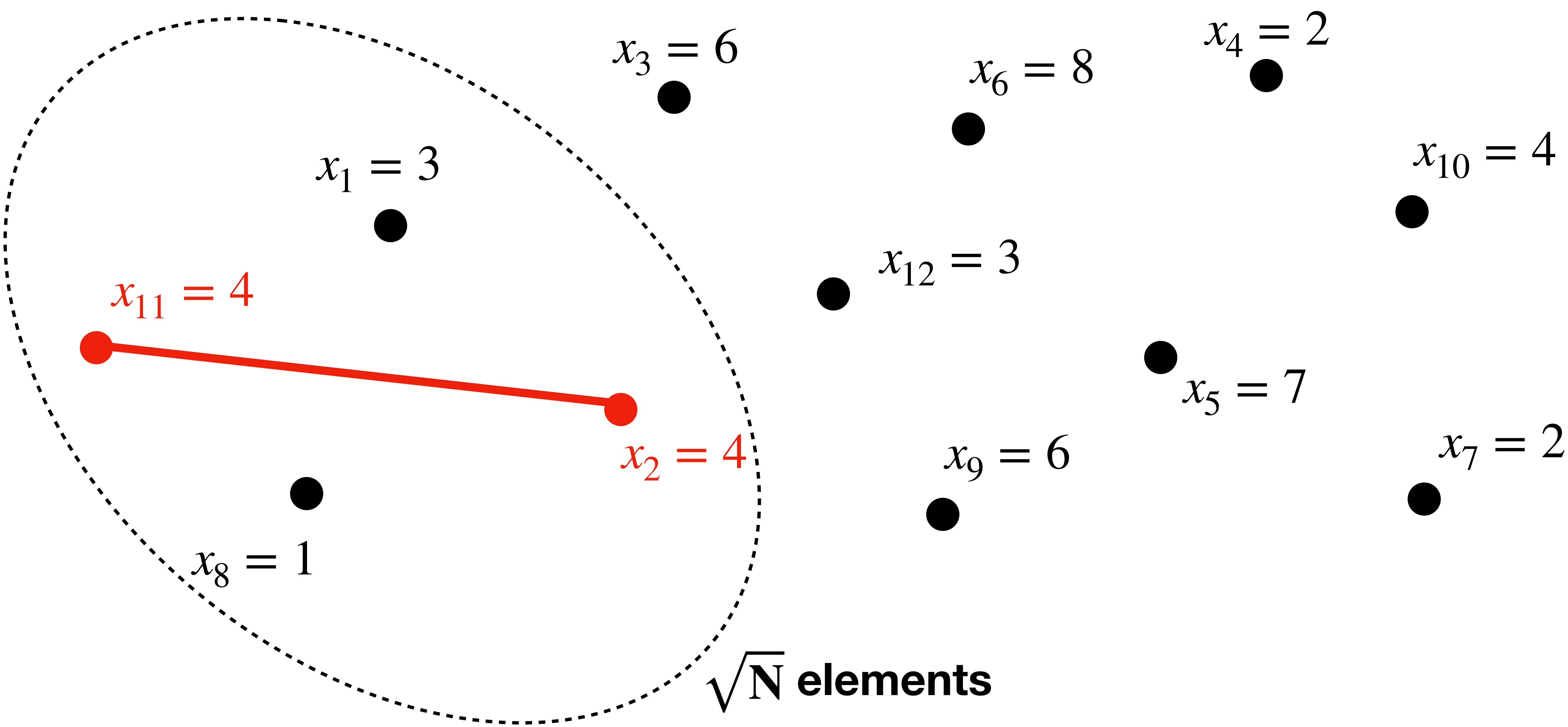


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$$T = O(\sqrt{N})$$

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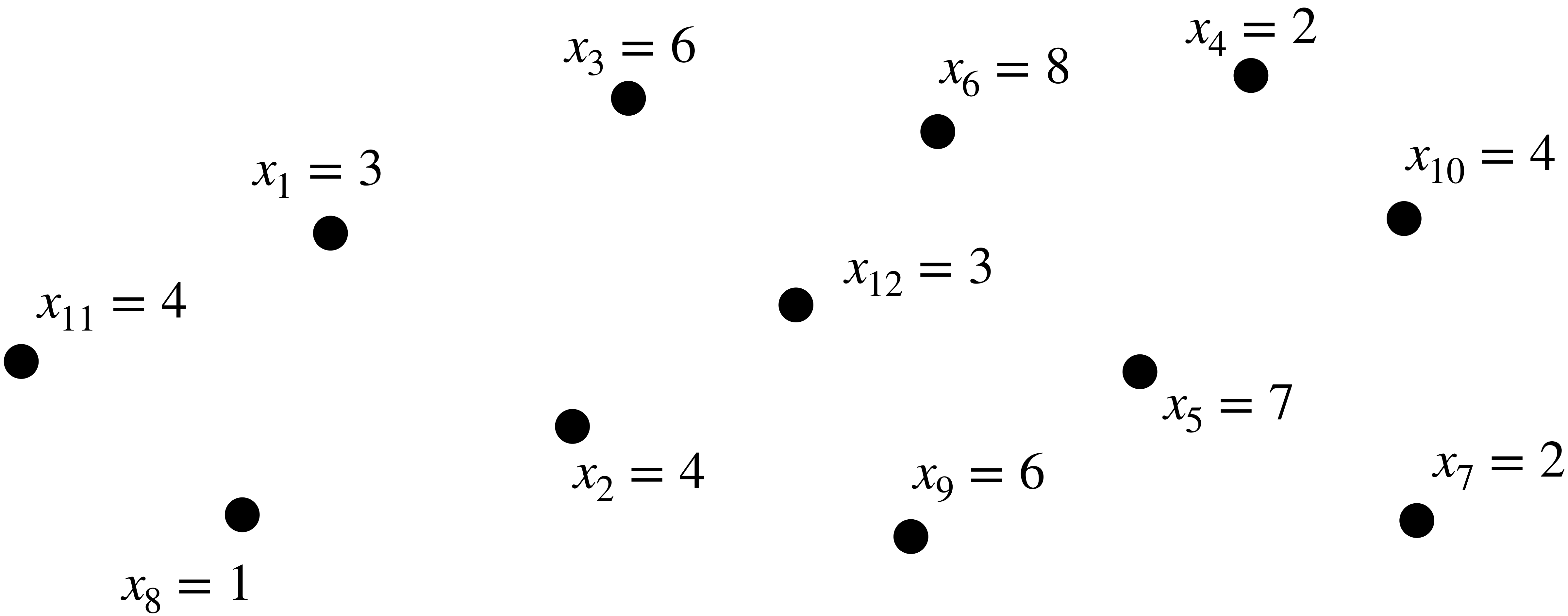


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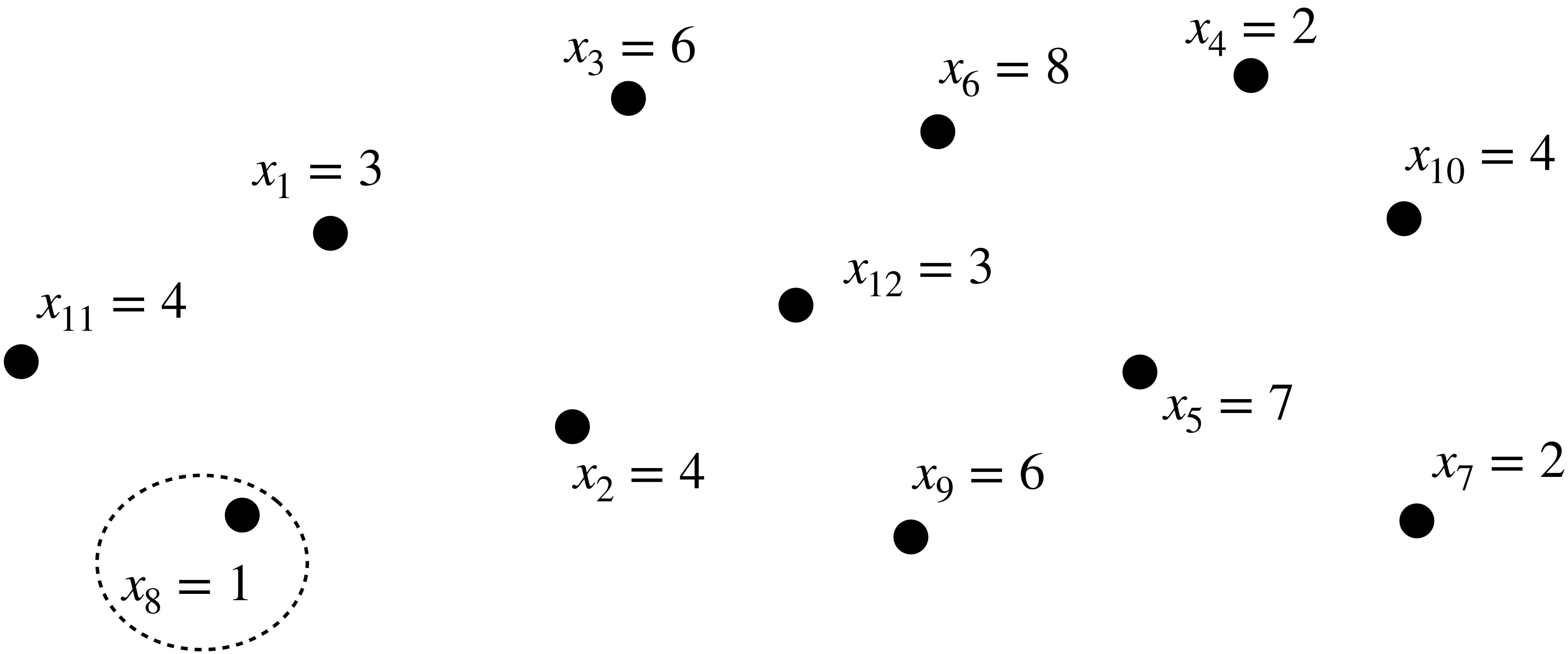


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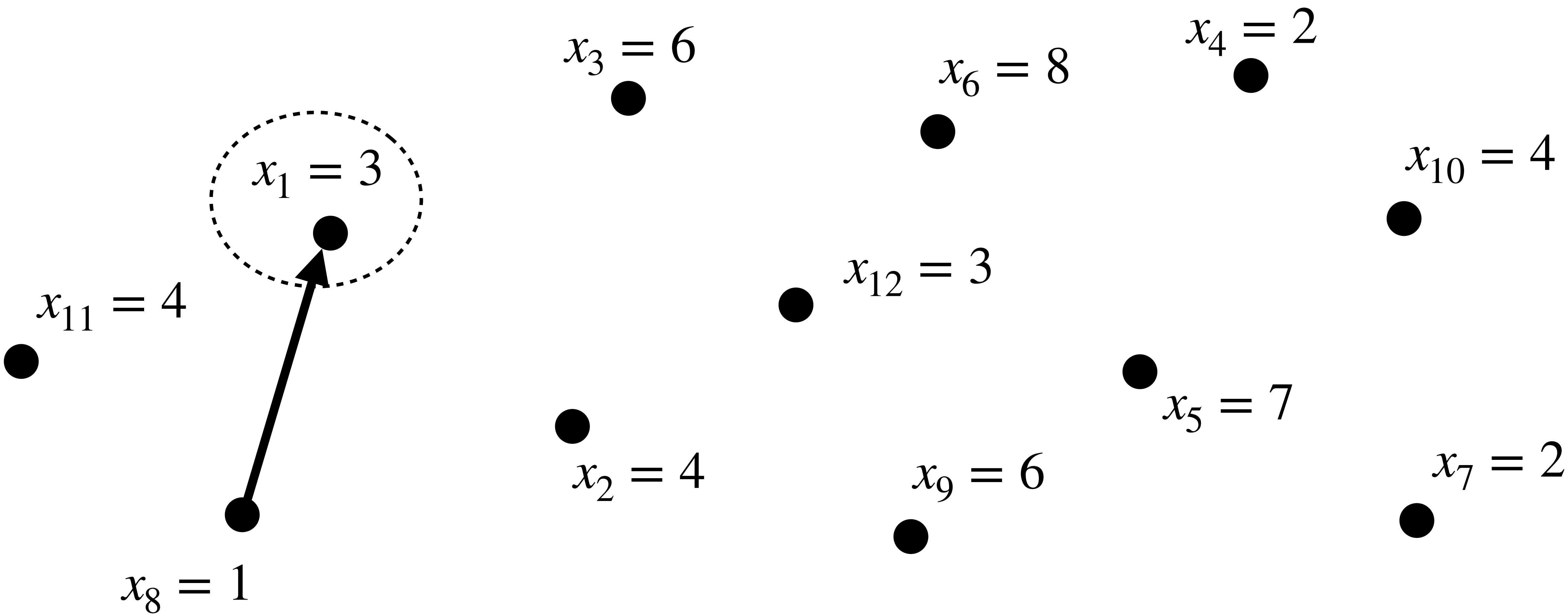


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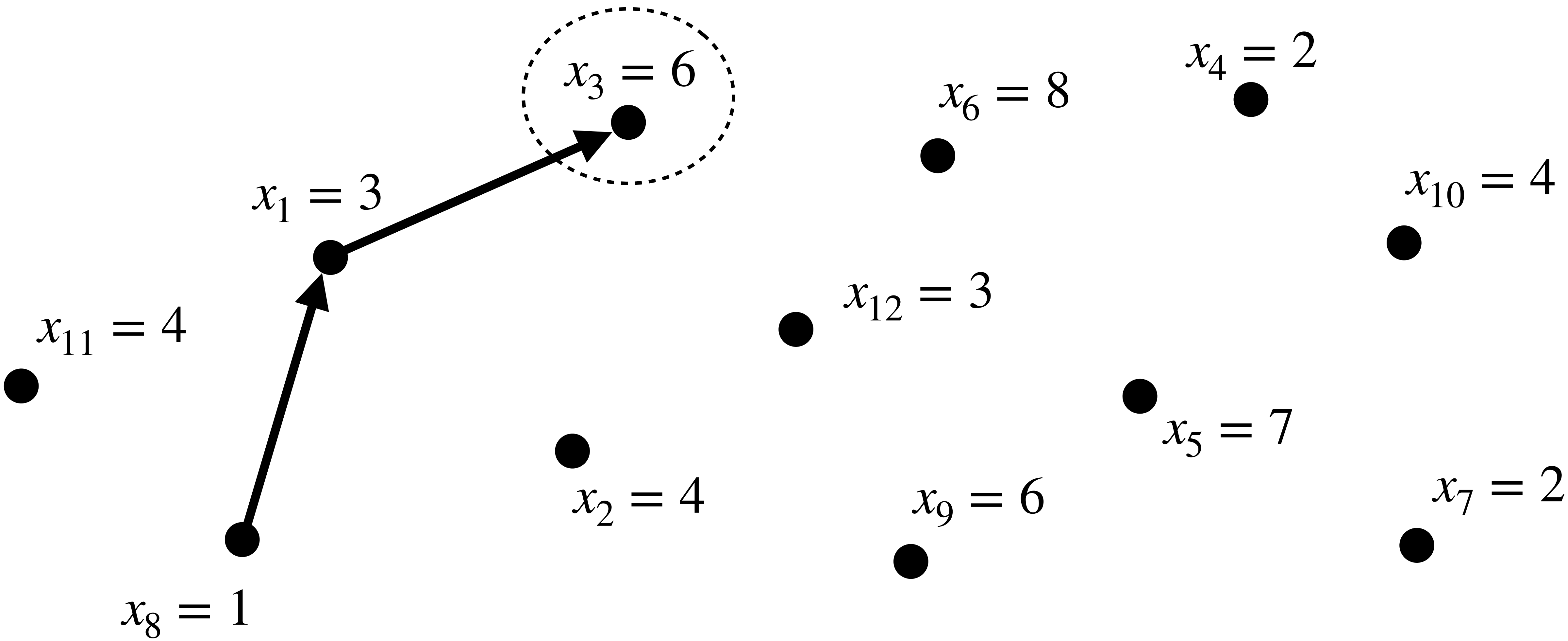


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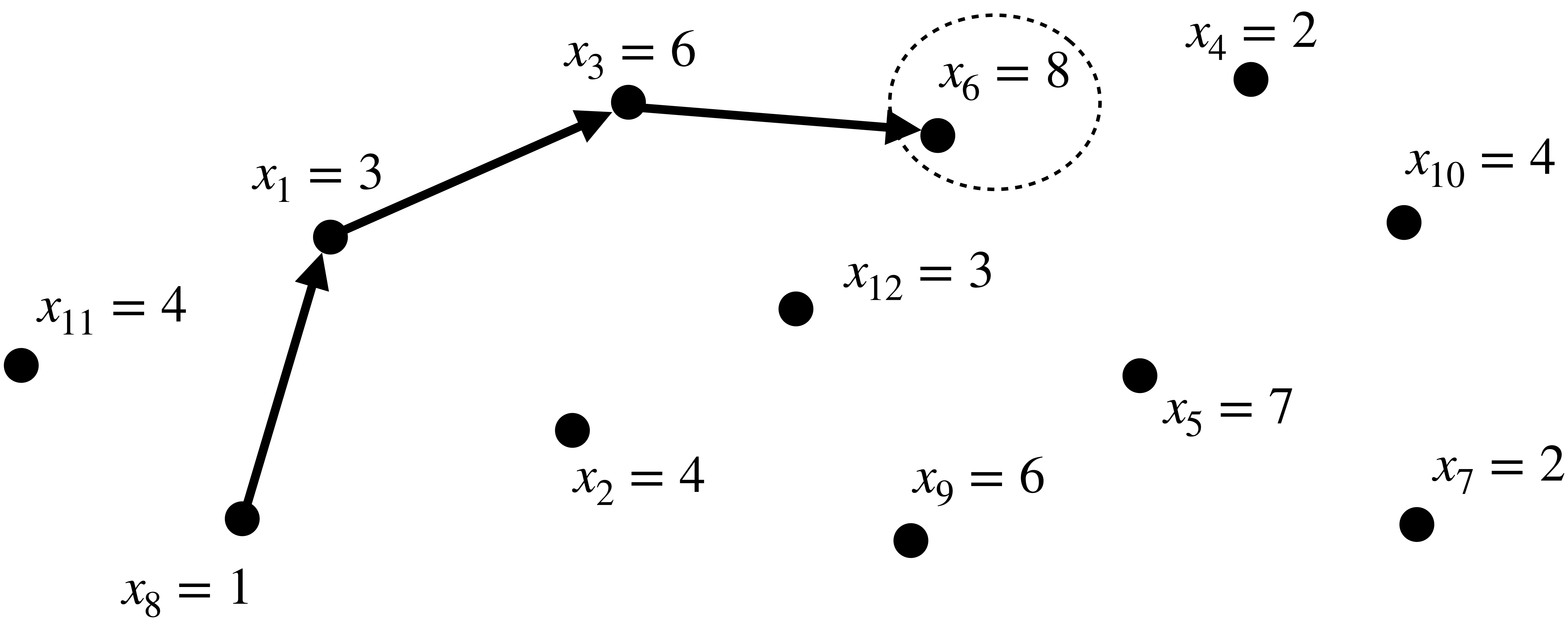


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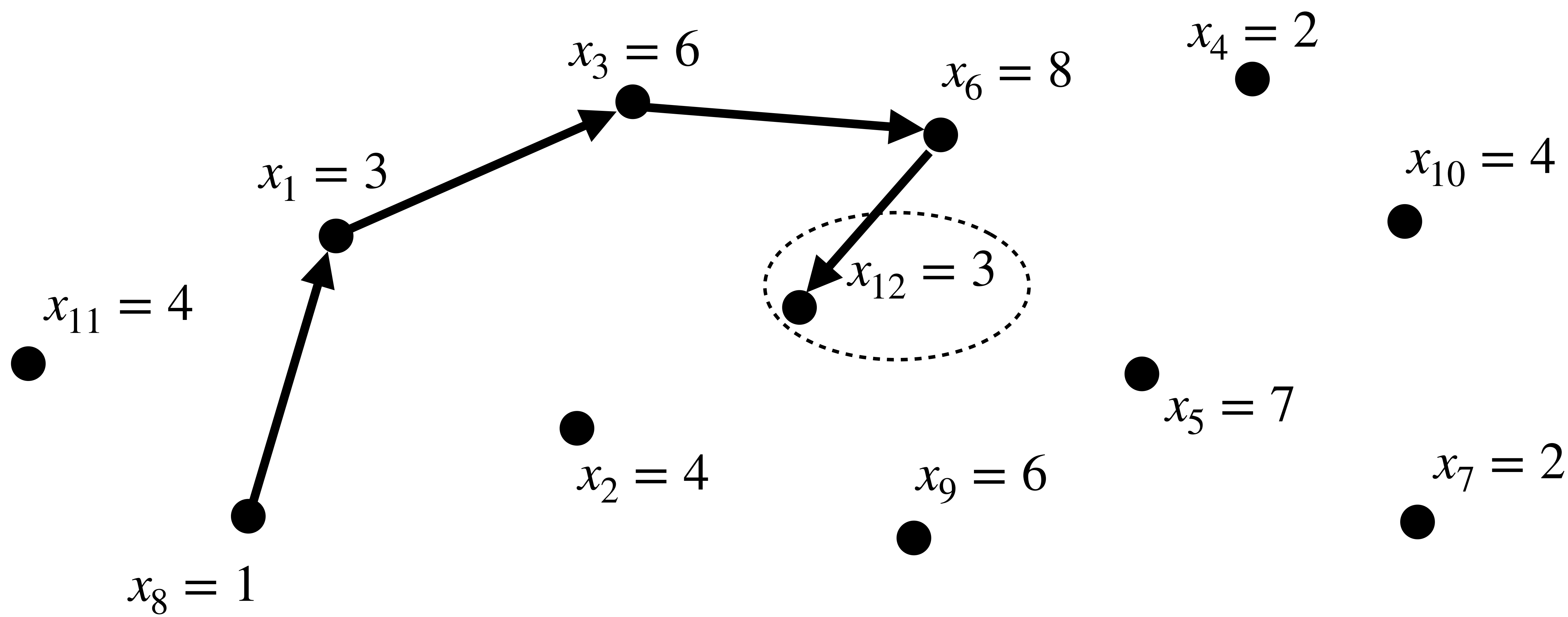


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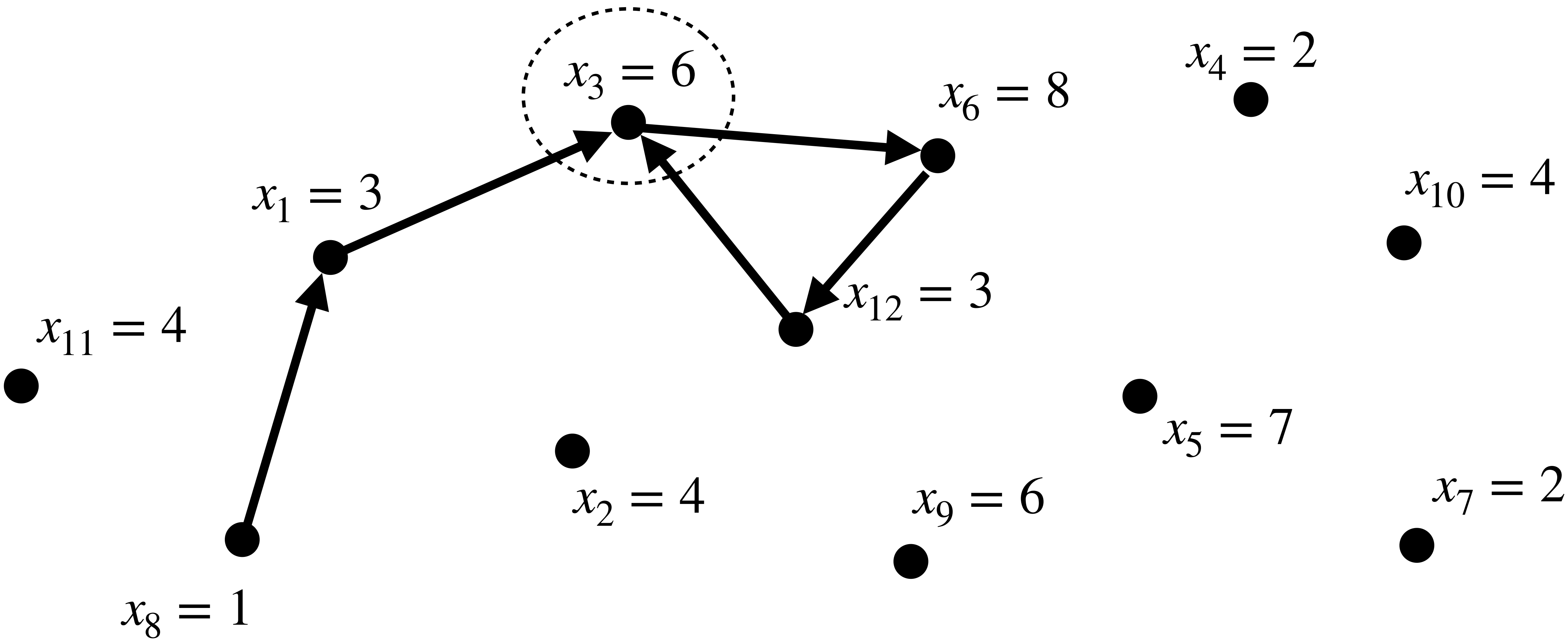


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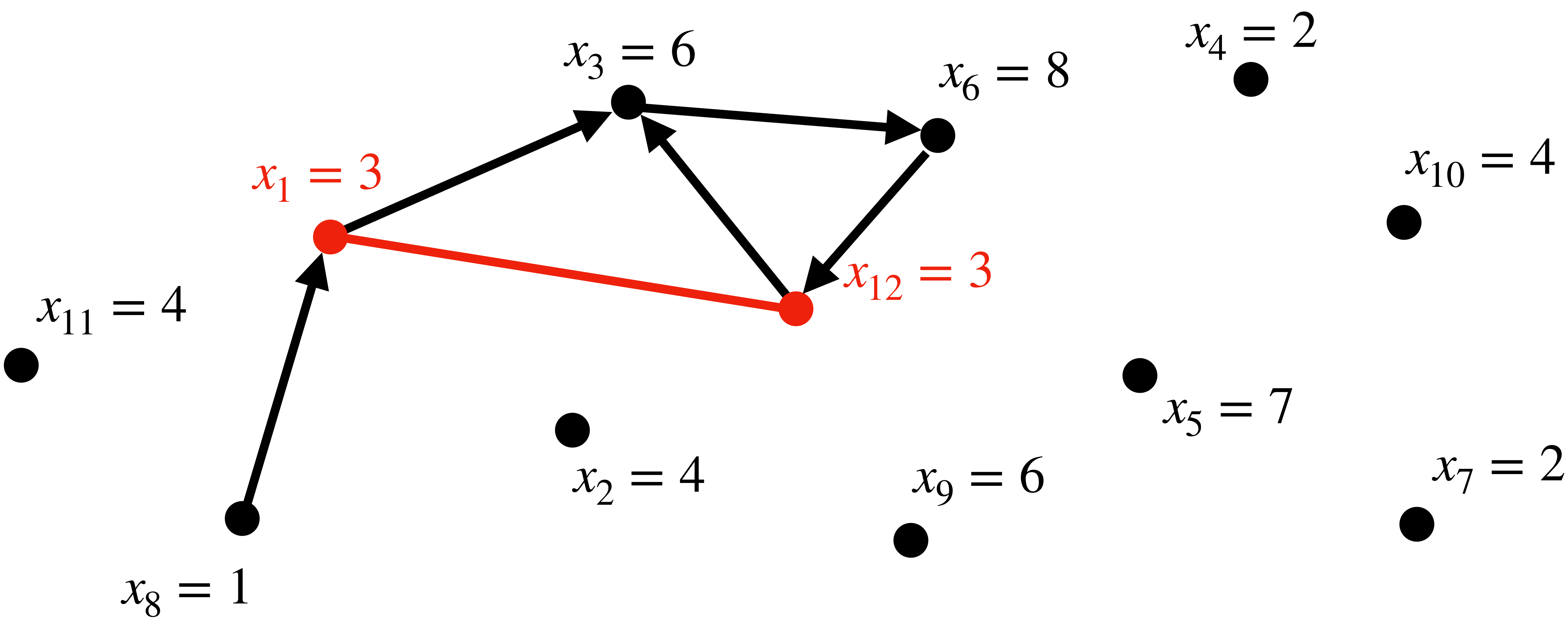


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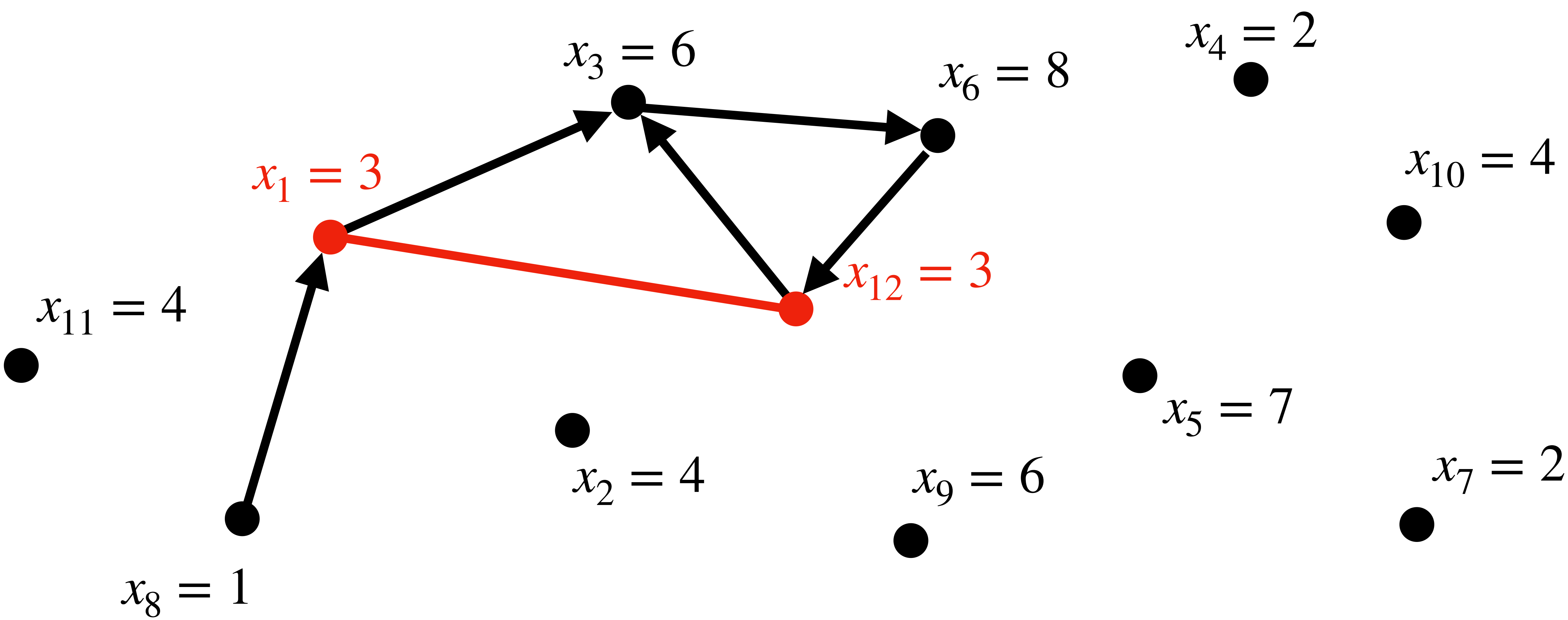


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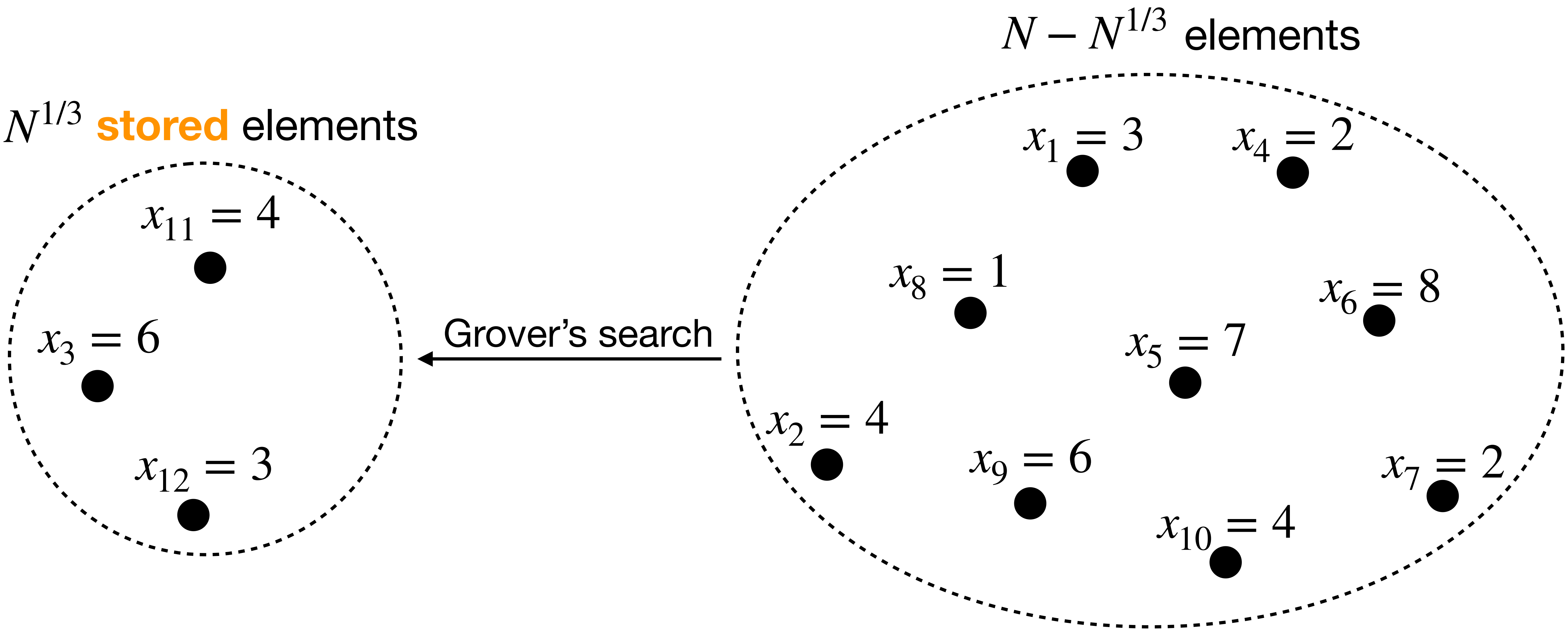


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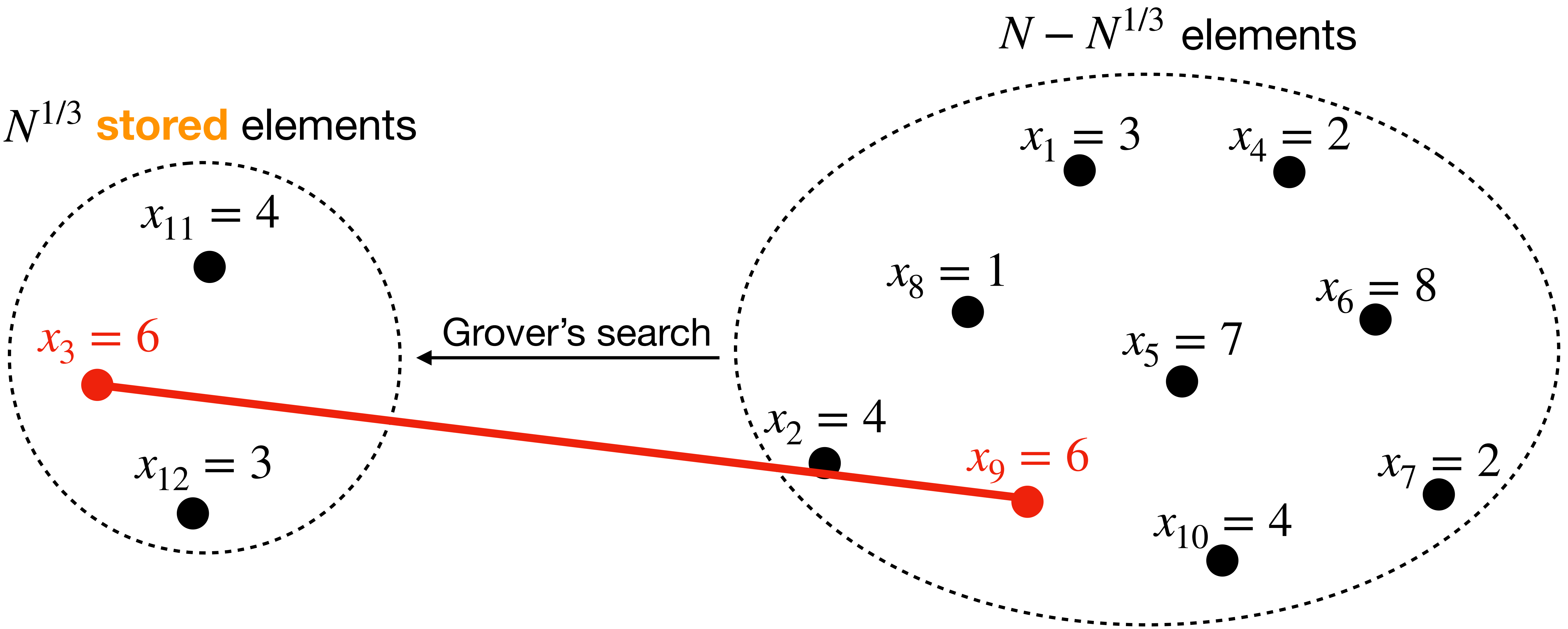


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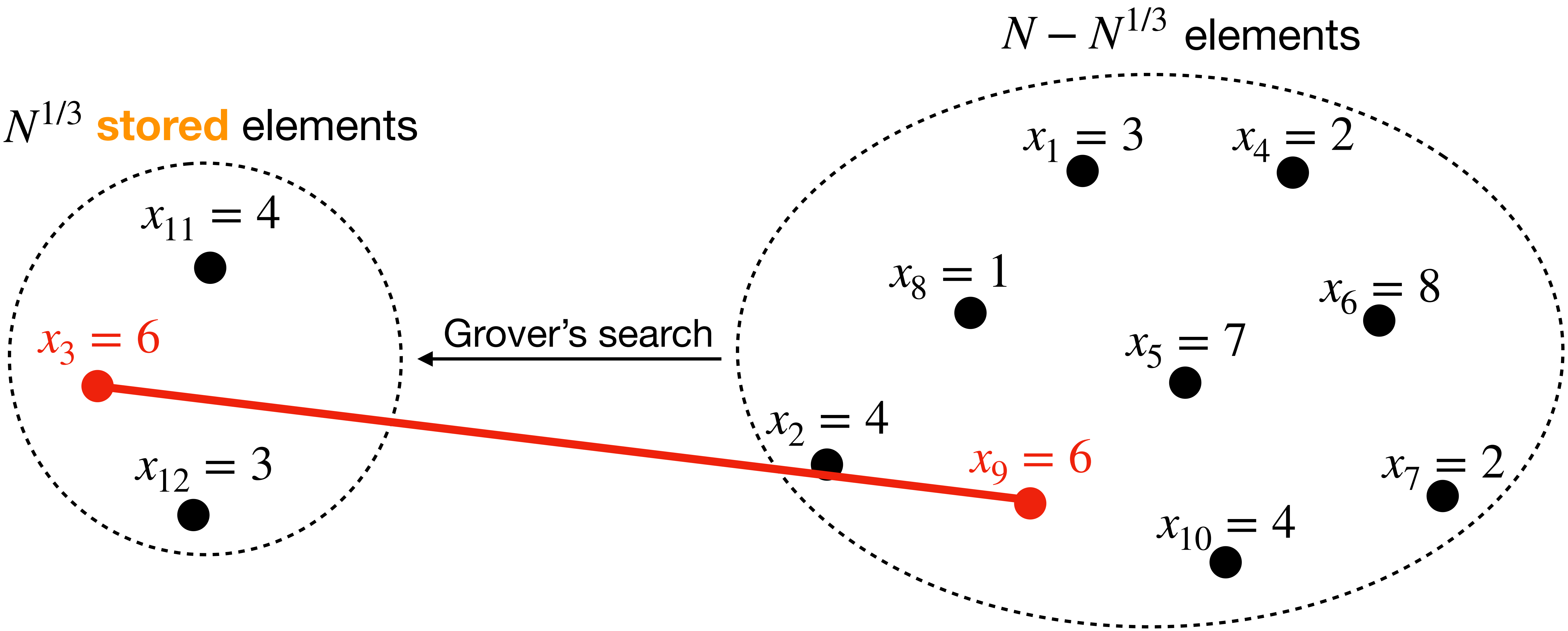


**Quantum BHT algorithm**

$$T = O\left(N^{1/3}\right)$$

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# Finding 1 collision pair



**Birthday attack +**
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**vs**

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The quantum BHT algorithm has a better time complexity, but a worst time-space tradeoff!

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A BHT attack on **SHA3-256** would require $S \approx 2^{256/3} \approx 2^{85}$ qubits!

# Finding 1 collision pair

| **Birthday attack + Floyd's cycle finding** | | **BHT algorithm** |
|---|---|---|
| $T = O(\sqrt{N})$ | **vs** | $T = O(N^{1/3})$ |
| $S = O(\log N)$ | | $S = O(N^{1/3})$ |

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A BHT attack on **SHA3-256** would require S ≈ $2^{256/3}$ ≈ $2^{85}$ qubits!

**Big open problem:** Is there a quantum algorithm with $T \leq o(\sqrt{N})$ and $S = O(\log N)$?

# Finding K collision pairs



| | Classical Tradeoff | Quantum Tradeoff |
|---|---|---|
| Upper bound | $T^2S \leq \tilde{O}(K^2N)$ when $\tilde{\Omega}(\log N) \leq S \leq \tilde{O}(K)$ <br> Parallel Collision Search [van Oorschot and Wiener'99] | $T^2S \leq \tilde{O}(K^2N)$ when $\tilde{\Omega}(\log N) \leq S \leq \tilde{O}(K^{2/3}N^{1/3})$ <br> Adaptation of the BHT algorithm |

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We conjecture: $T^2S \geq \tilde{\Omega}(K^2N)$

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  → If true, we show that it would implies $T^2S \geq \tilde{\Omega}(N^2)$ for **Element Distinctness.**

3

# Lower Bounds by Recording Queries

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**K Collisions**

Finding **K collisions** in $x \sim [N]^N$ with success probability at least $2^{-O(K)}$ requires time $T \geq \Omega\left(K^{2/3}N^{1/3}\right)$.

$\Rightarrow$ $T^3 S \geq \tilde{\Omega}(K^3 N)$ for K-Collision Pairs Finding

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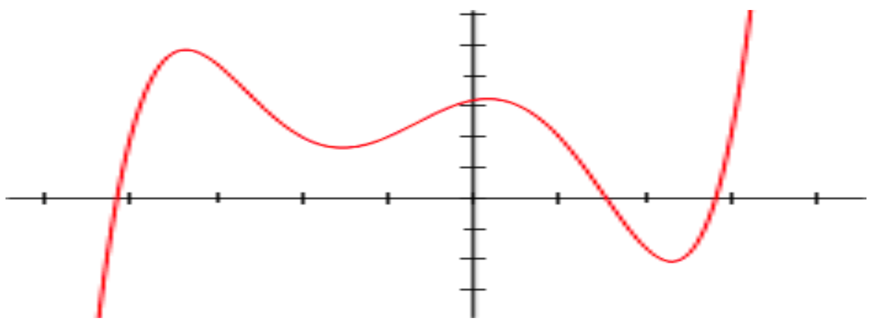

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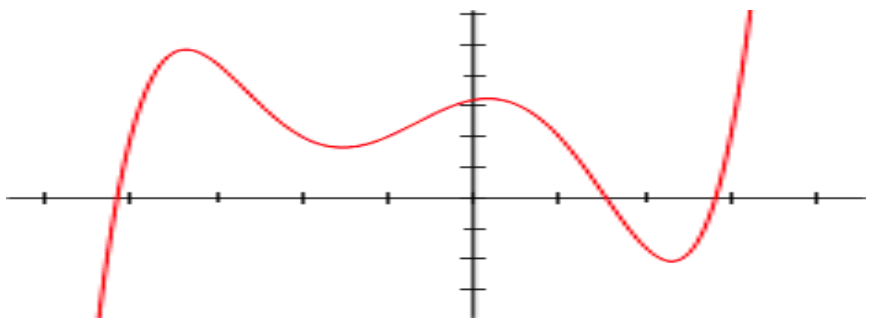

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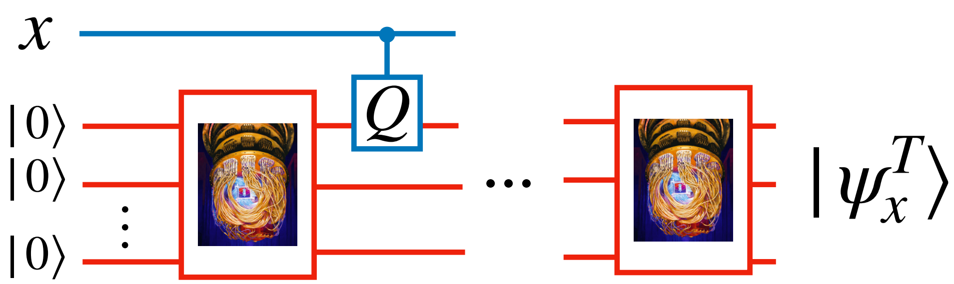

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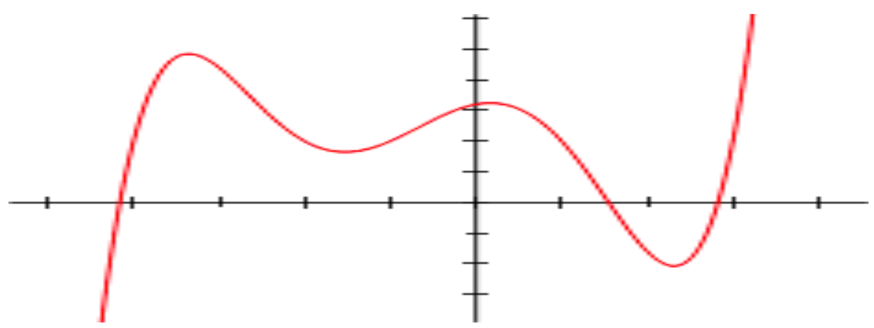

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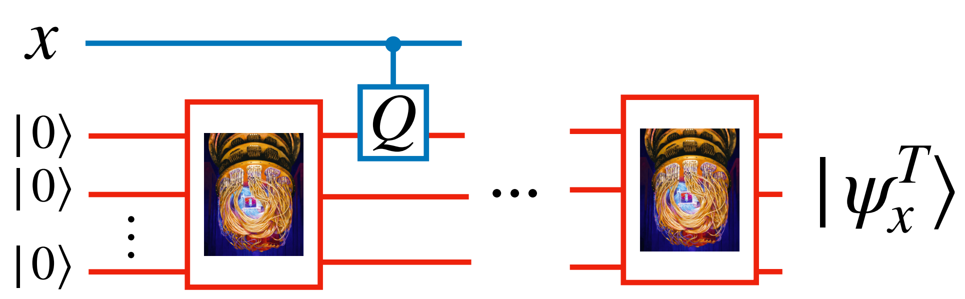


**Both methods are often difficult to use in practice:**

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Coppersmith-Rivlin's bound + Extremal properties of Chebyshev polynomials.

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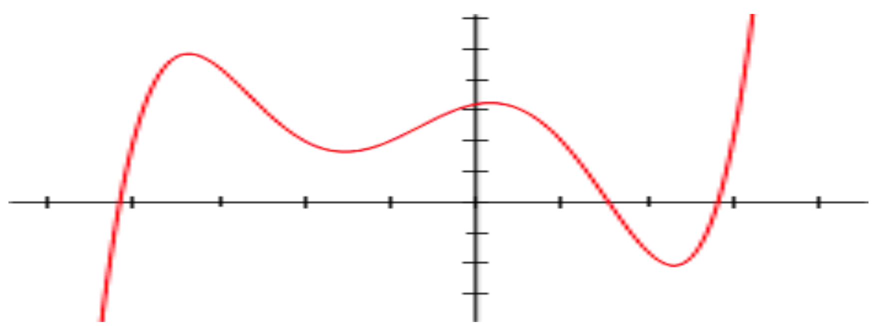

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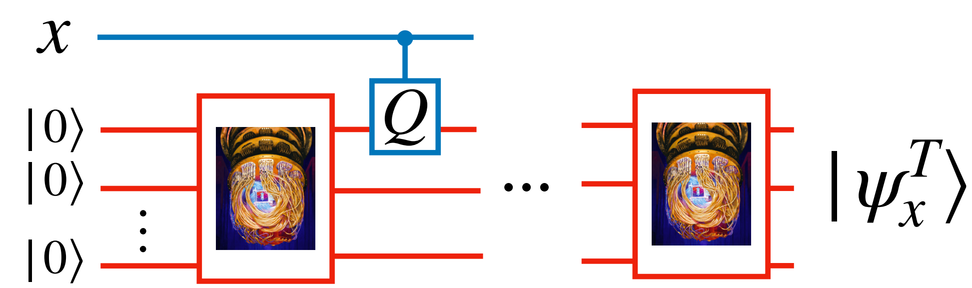


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**A simpler and more intuitive method?**

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(The un-recorded positions can only be guessed, with success $\leq (K/N)^{K/2} \leq 2^{-\Omega(K)}$).

# Recording of Quantum Queries



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**[Zhandry'19]:**

- A quantum "recording technique" that works when the input $x_1, \ldots, x_N$ is sampled from the **uniform distribution** on $[M]^N$.
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**Our contribution:**

- We generalize Zhandry's technique to the case where $x_1, \ldots, x_N$ is sampled from any **product distribution $D_1 \otimes \ldots \otimes D_N$** on $[M]^N$.
- We simplify the framework and the analysis of the method.

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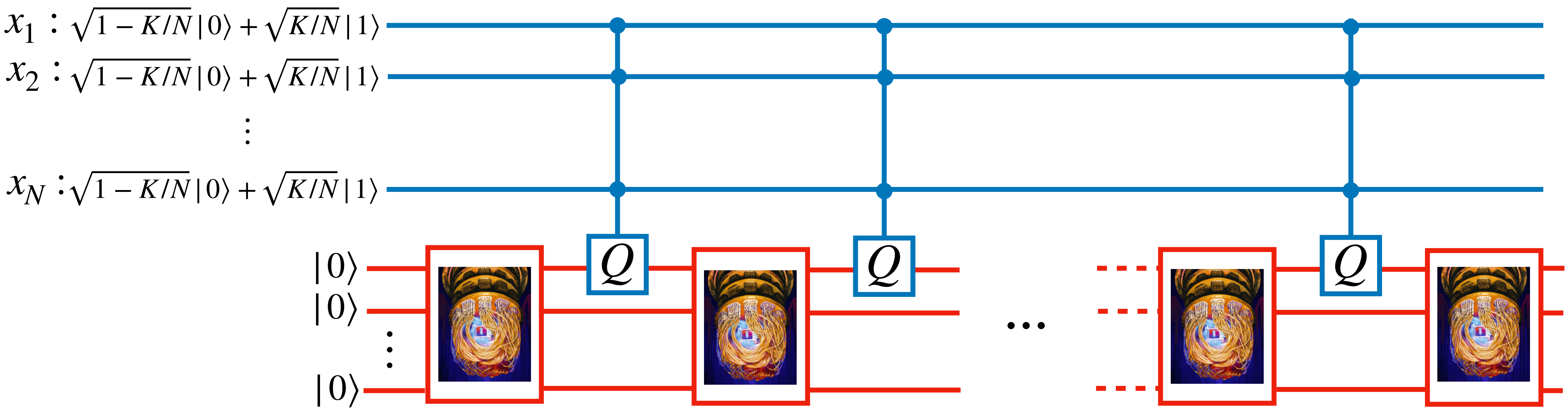
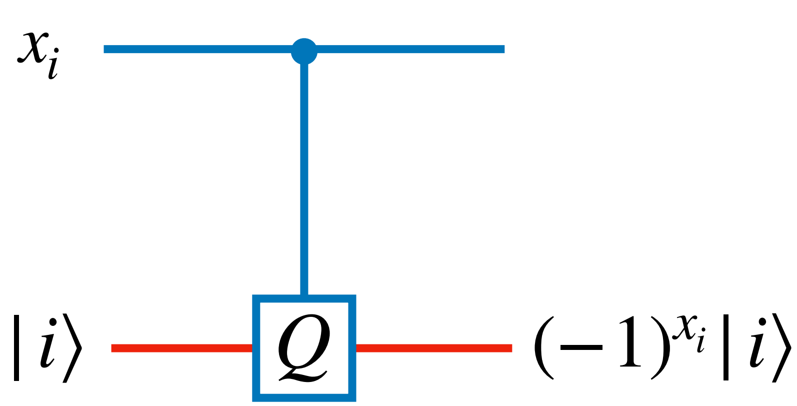



## Query Operator



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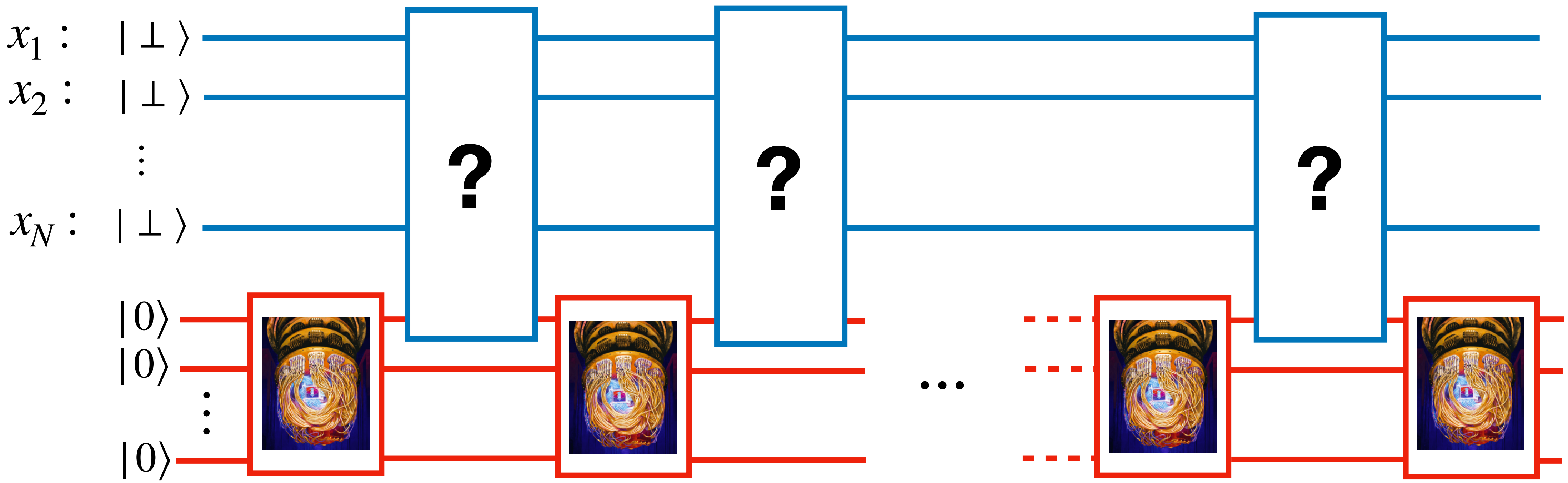


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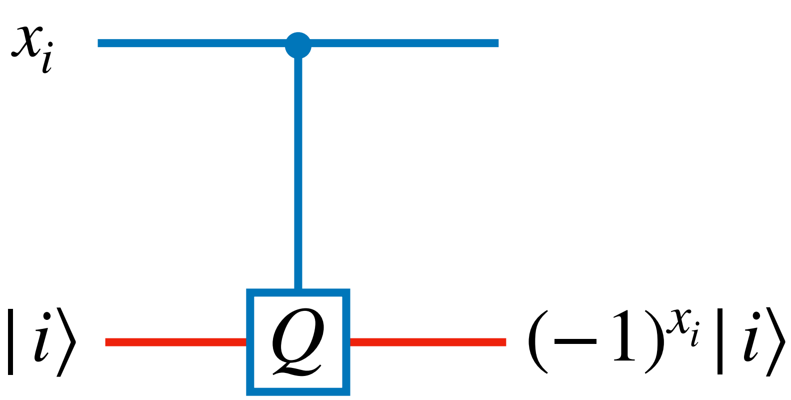

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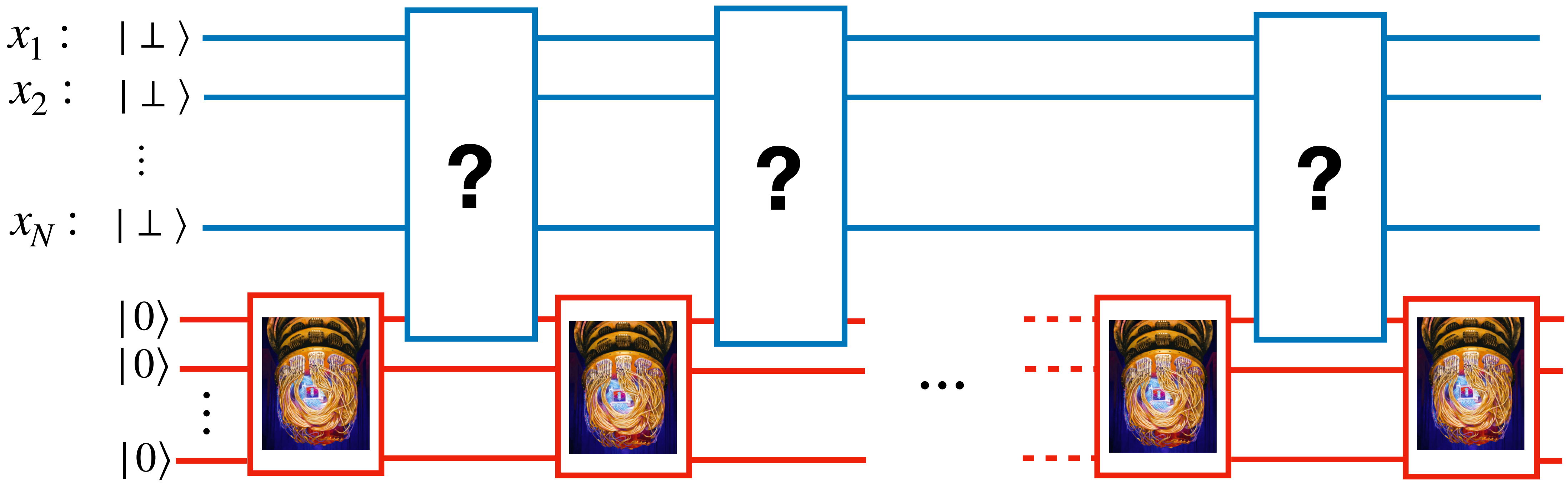


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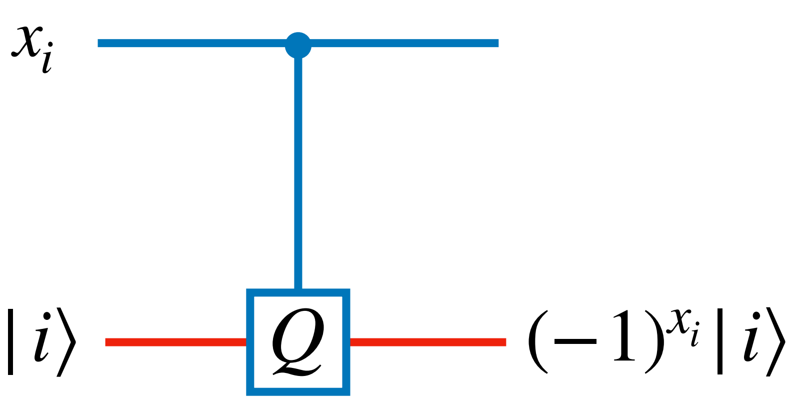

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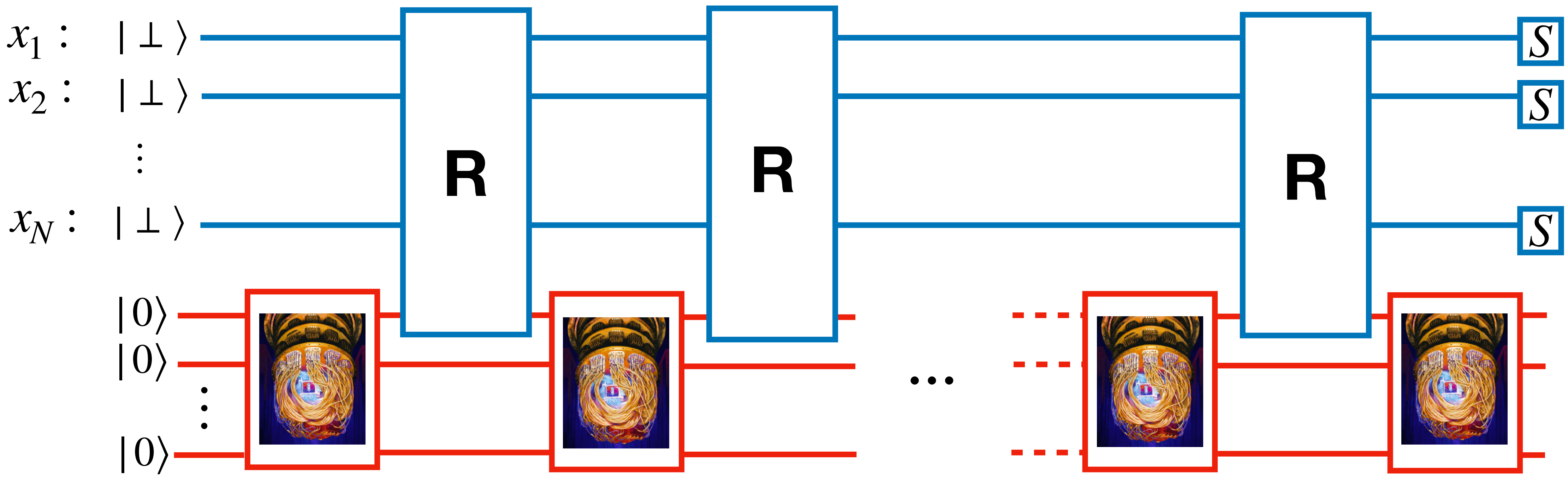


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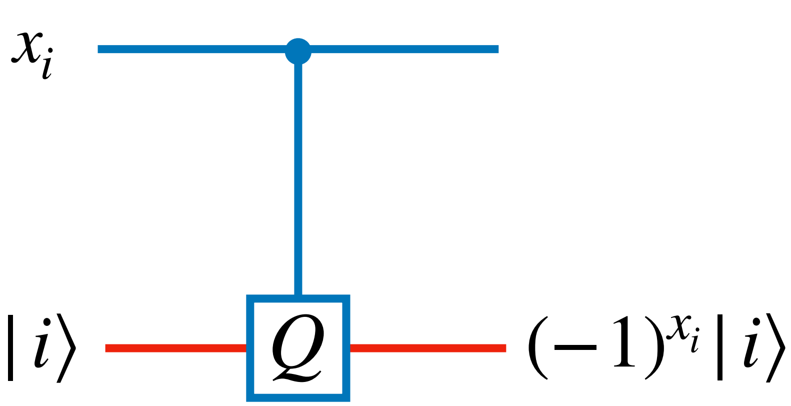


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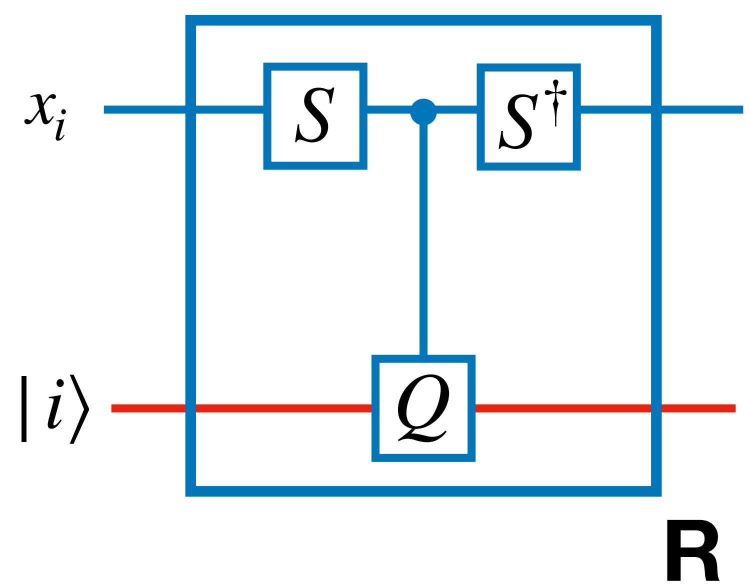


(where $(-1)^\perp = 1$)

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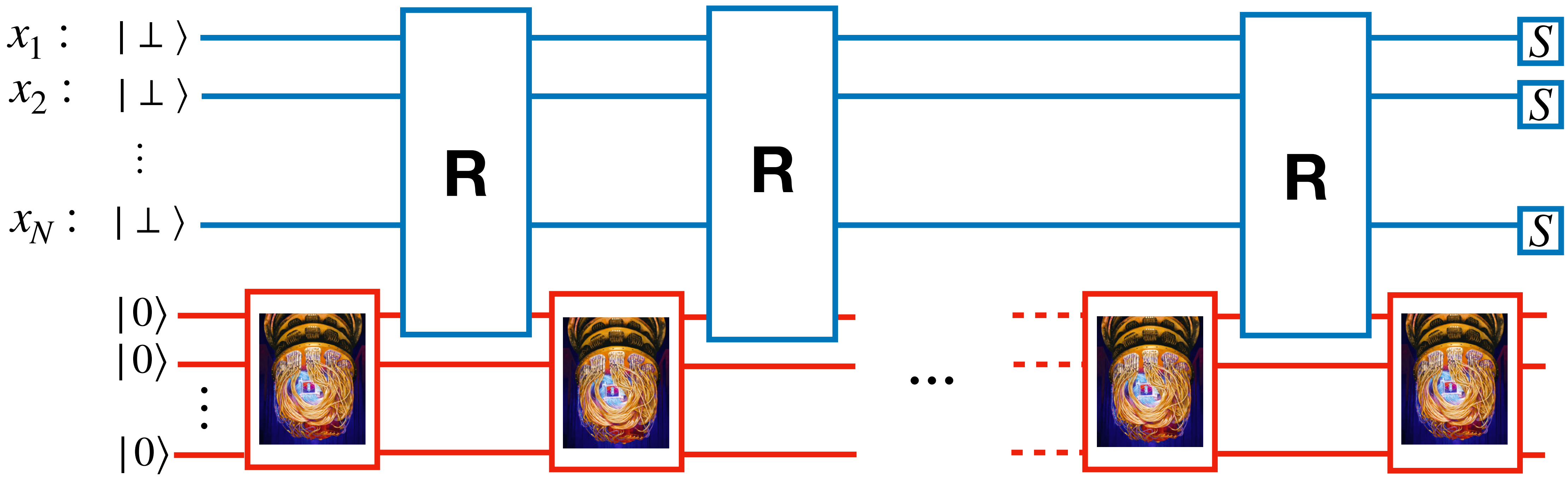


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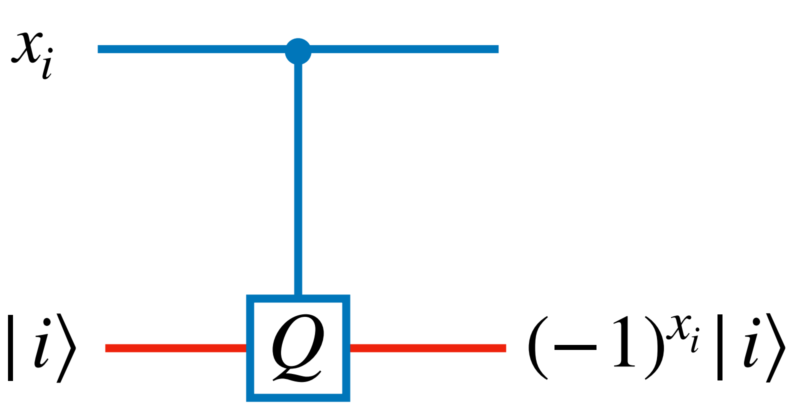


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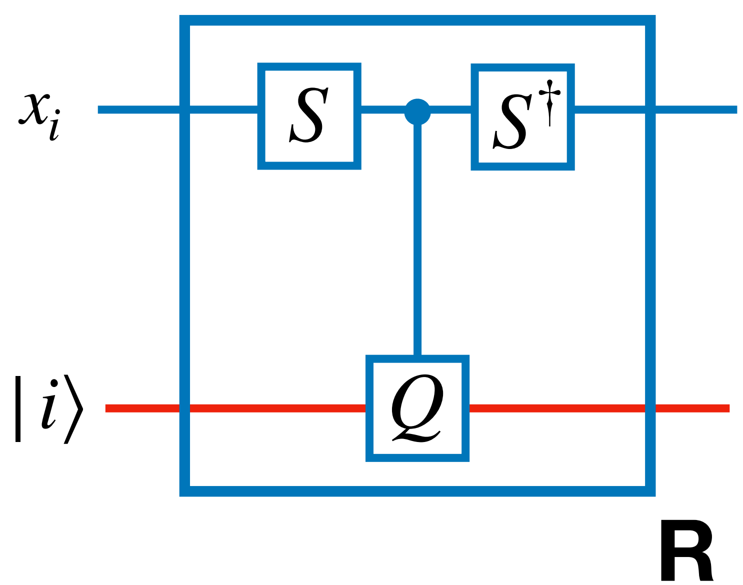


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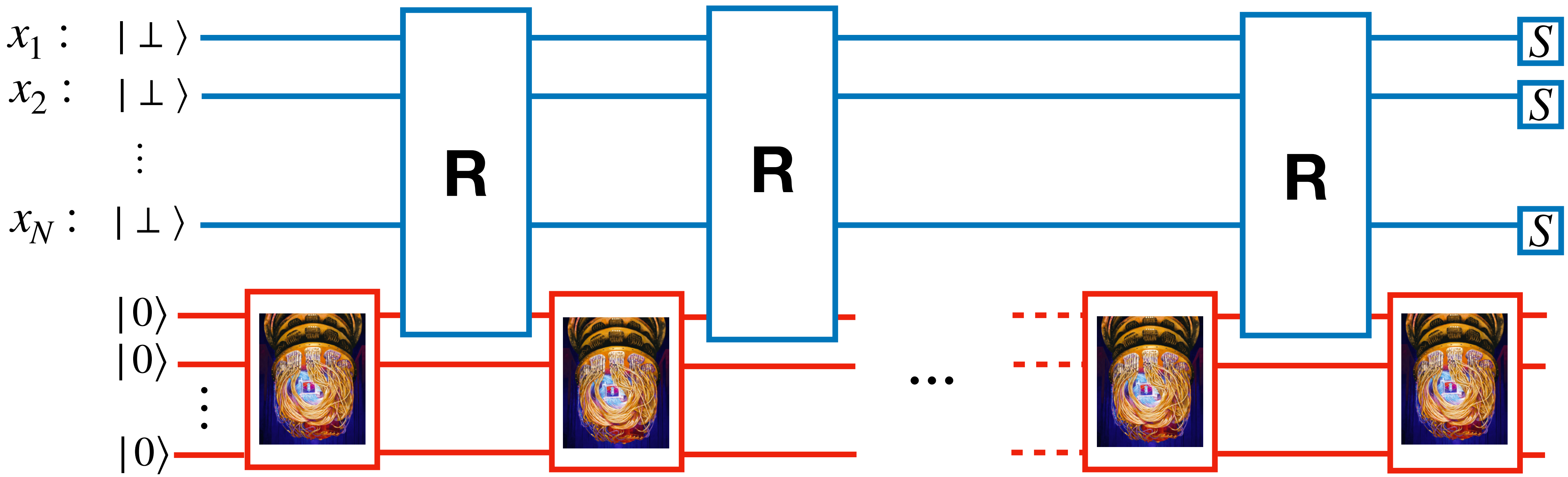


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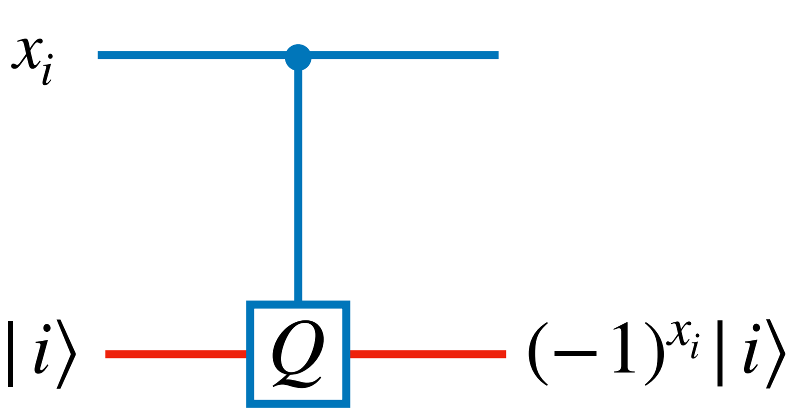


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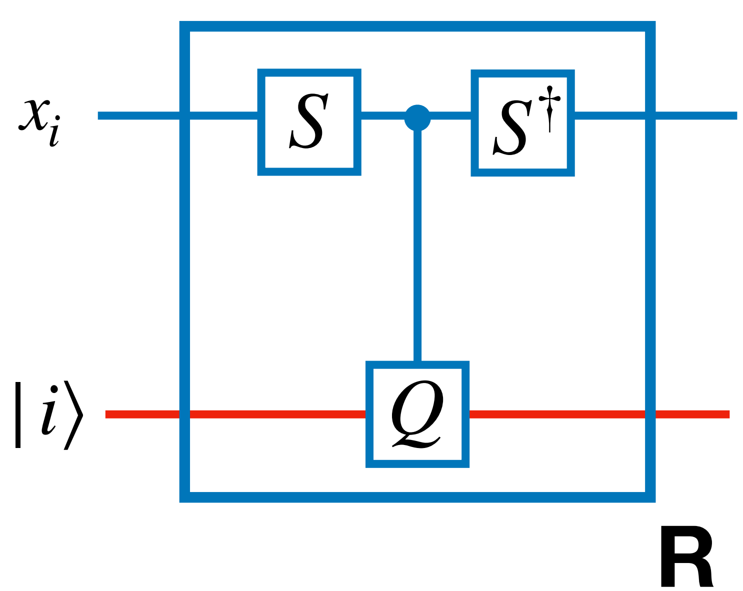


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- ✓ We show that it "records" the 1's.

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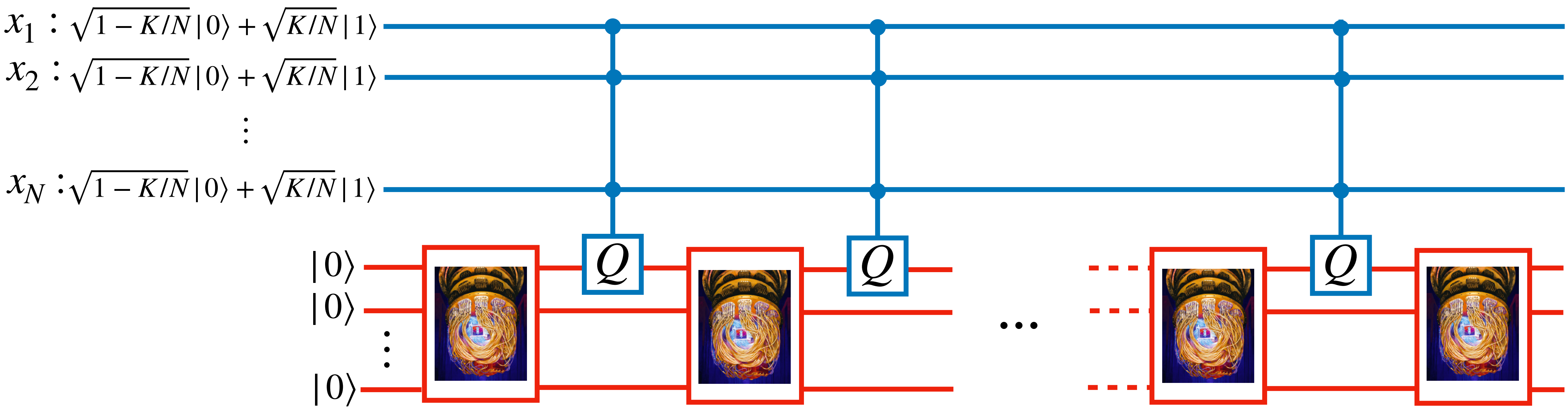


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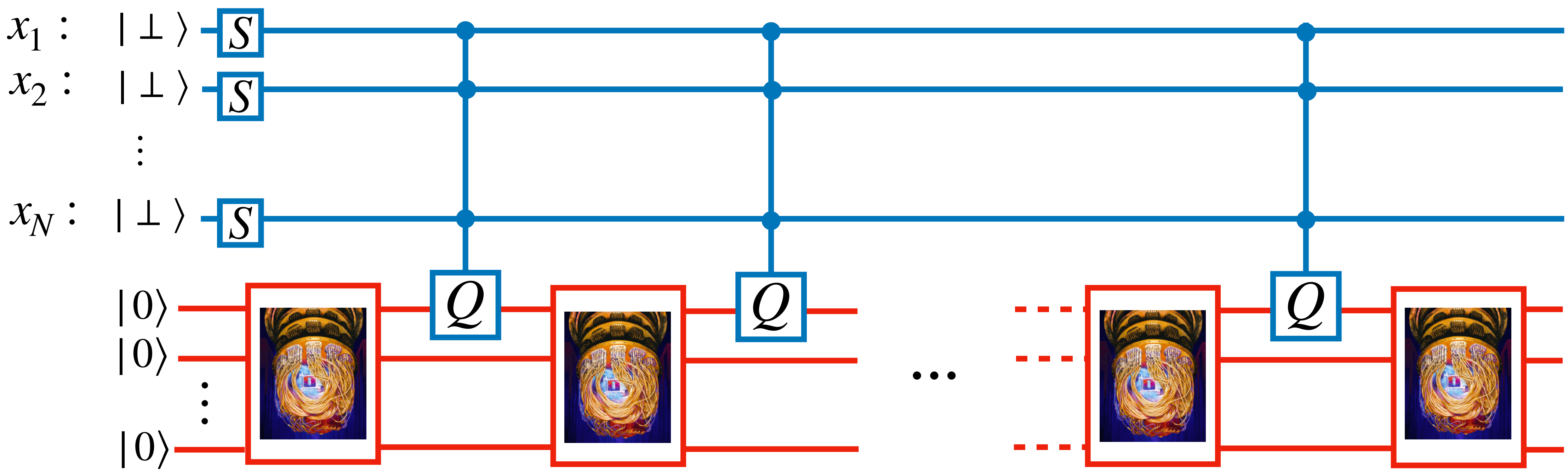


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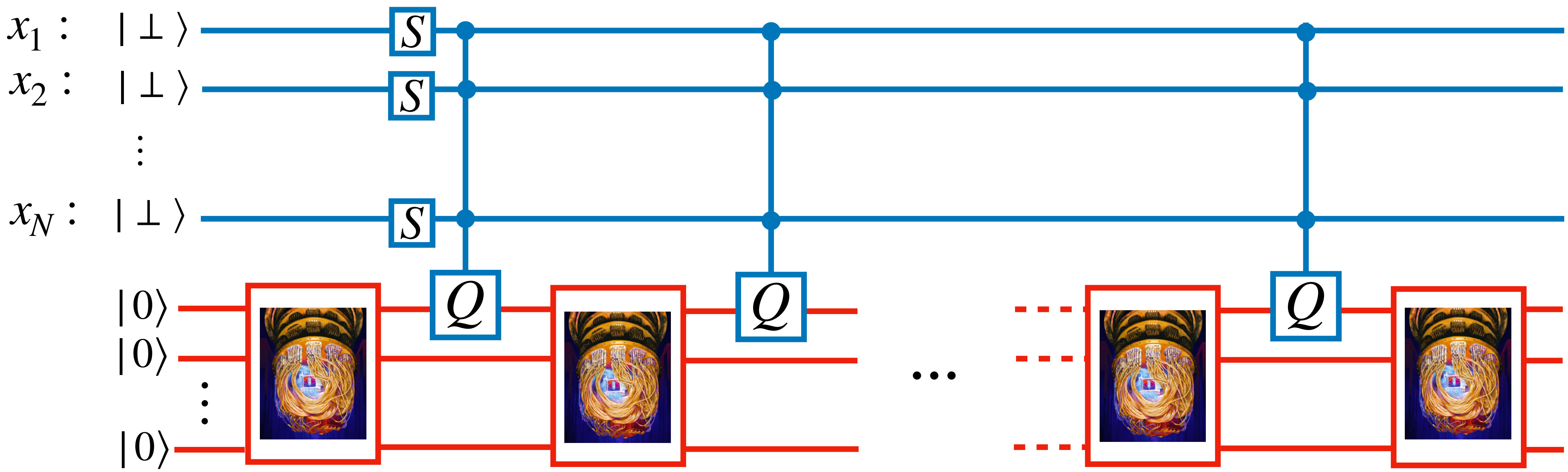


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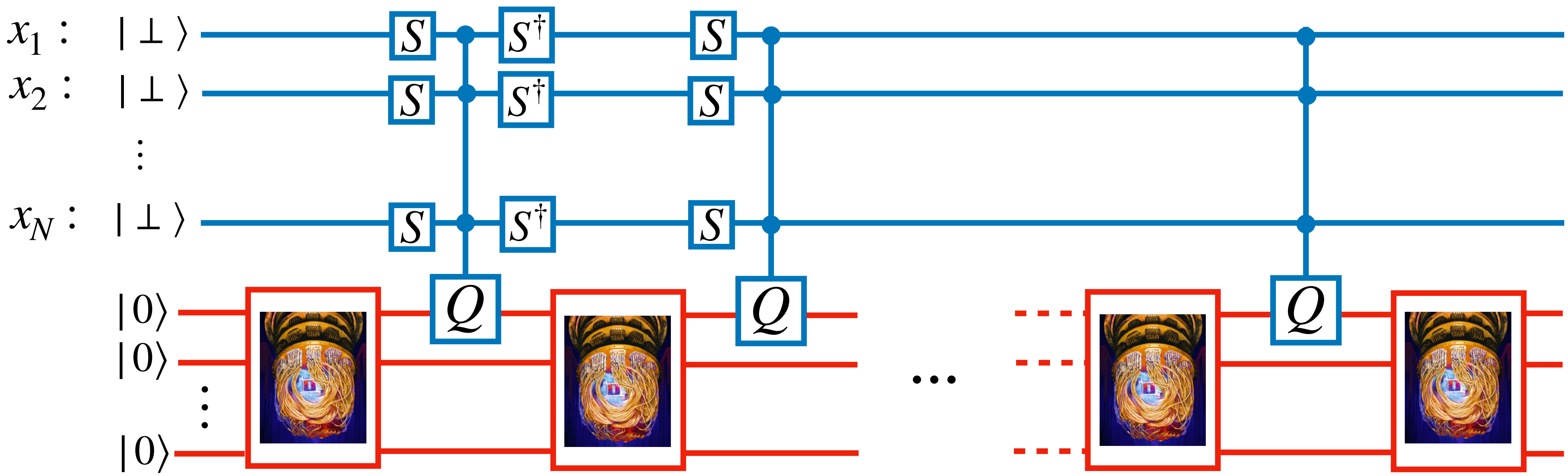


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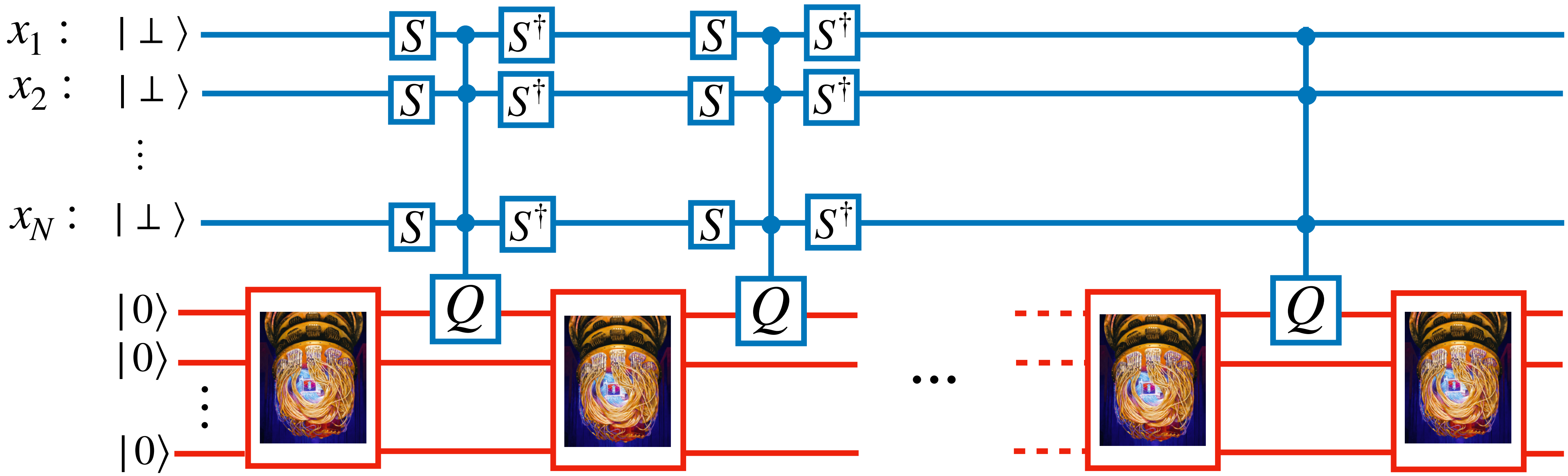


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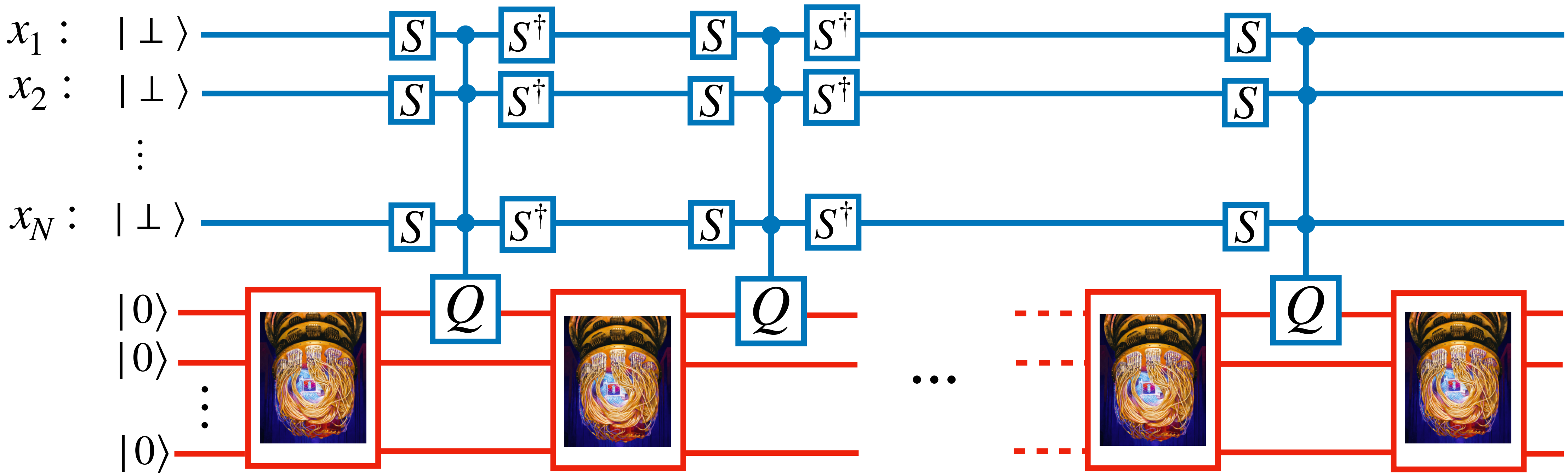


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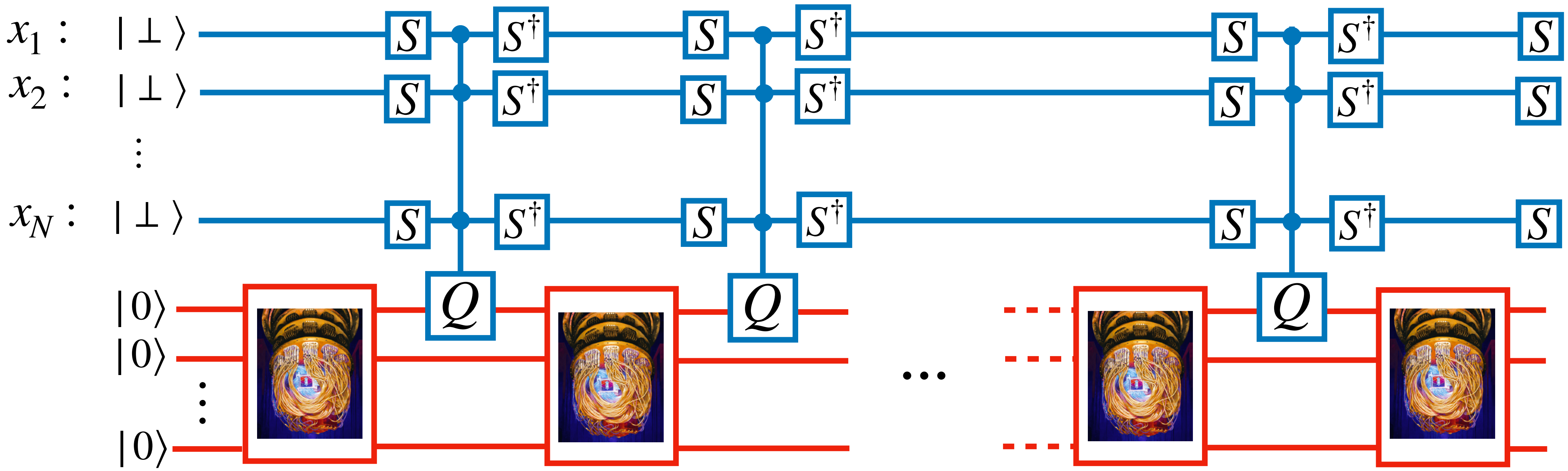


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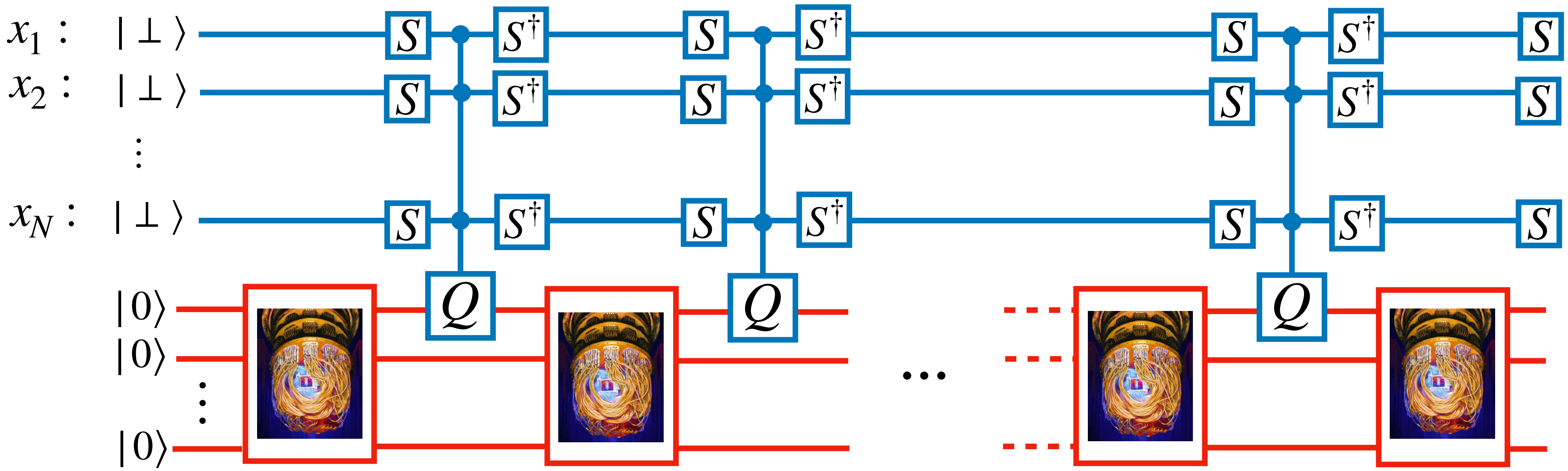


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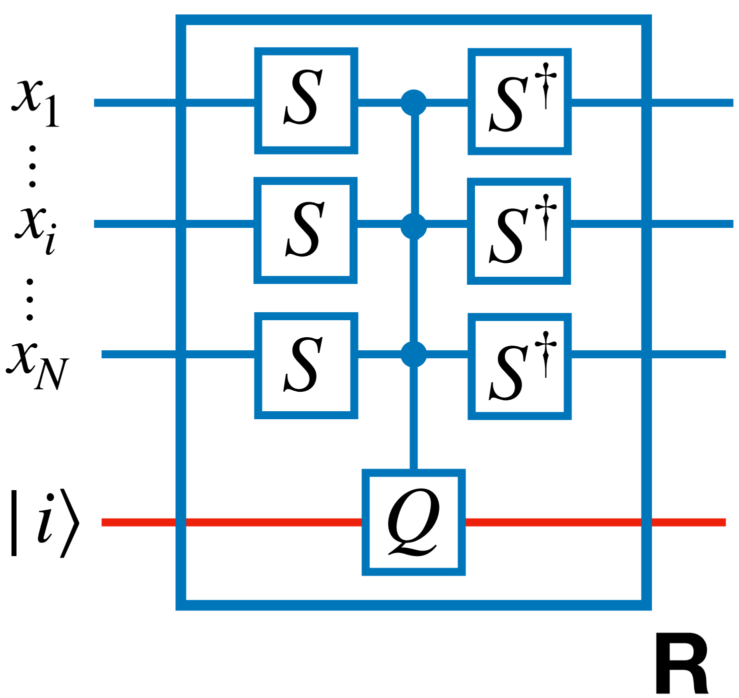

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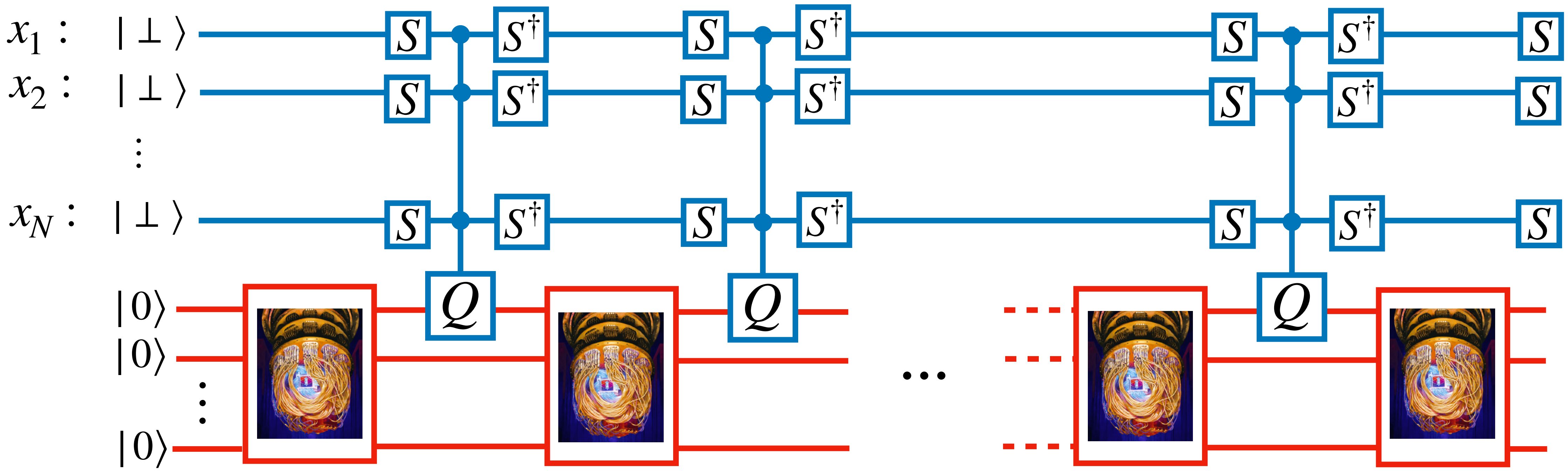


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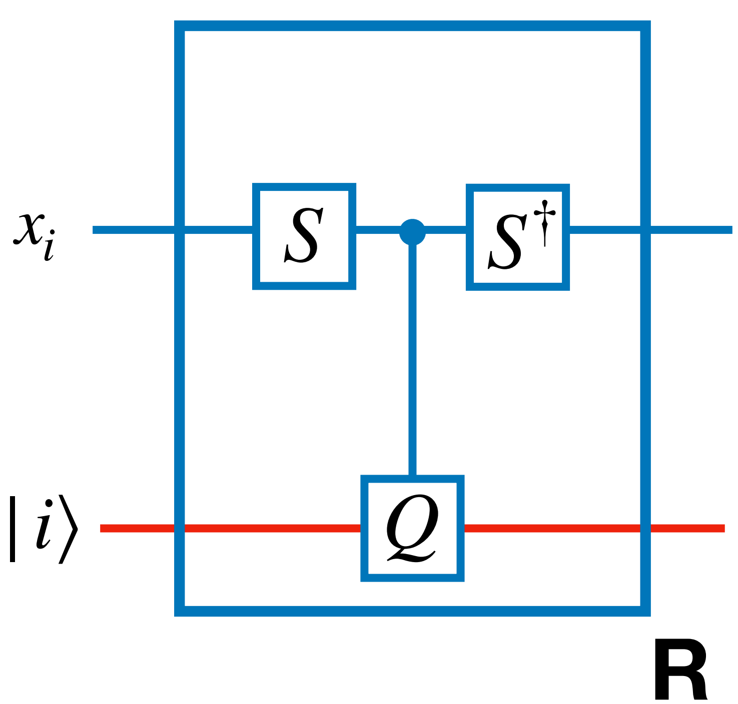


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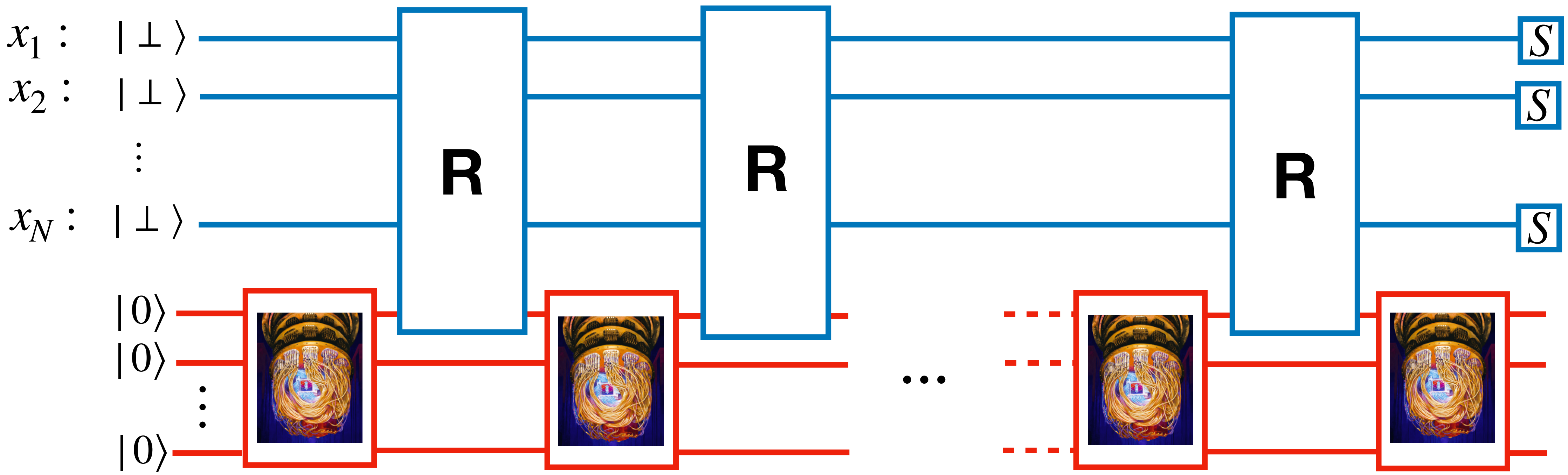


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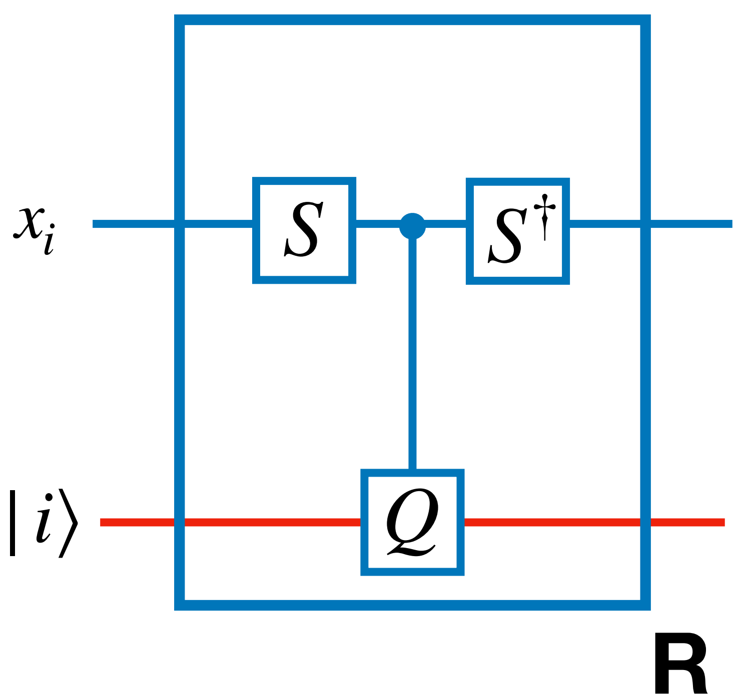


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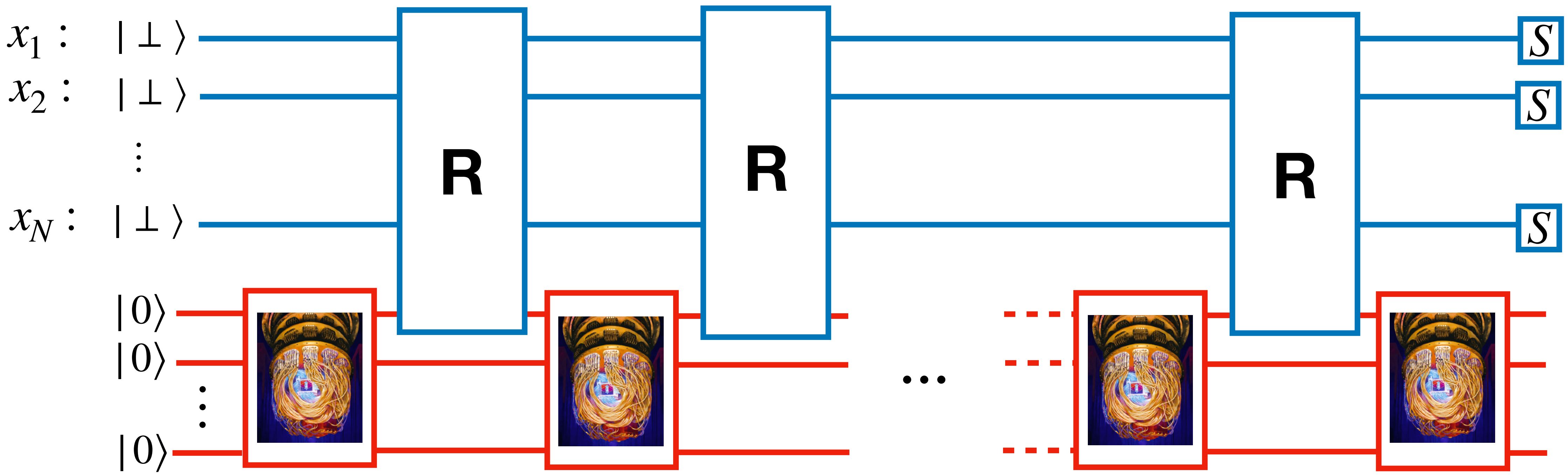


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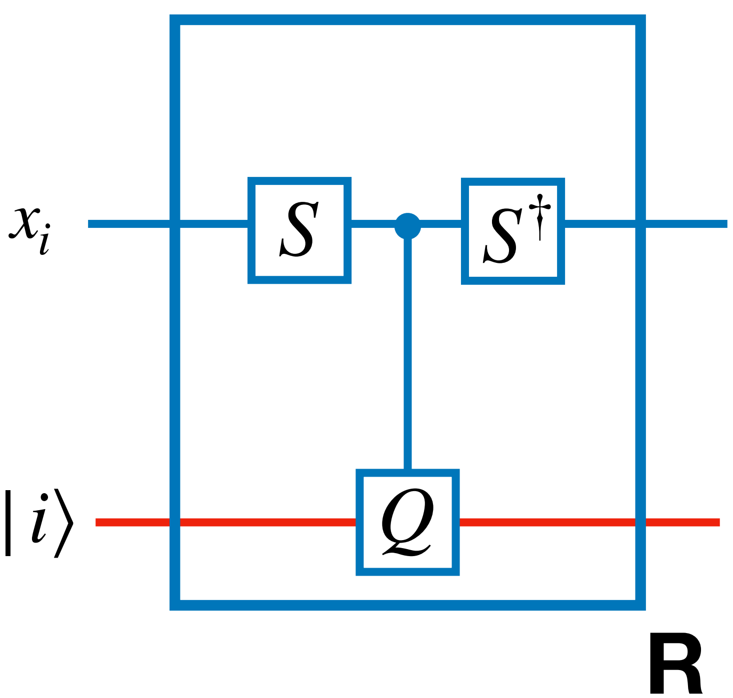
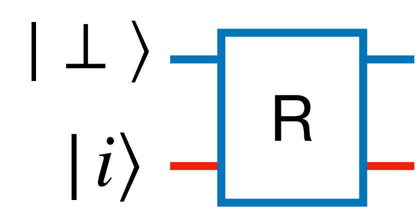
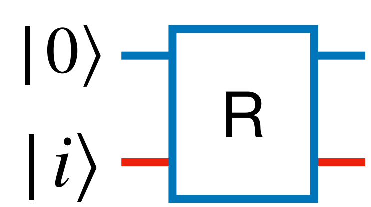
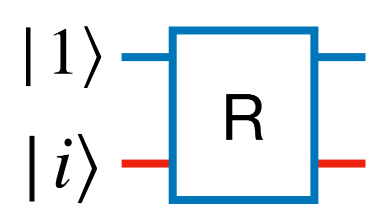


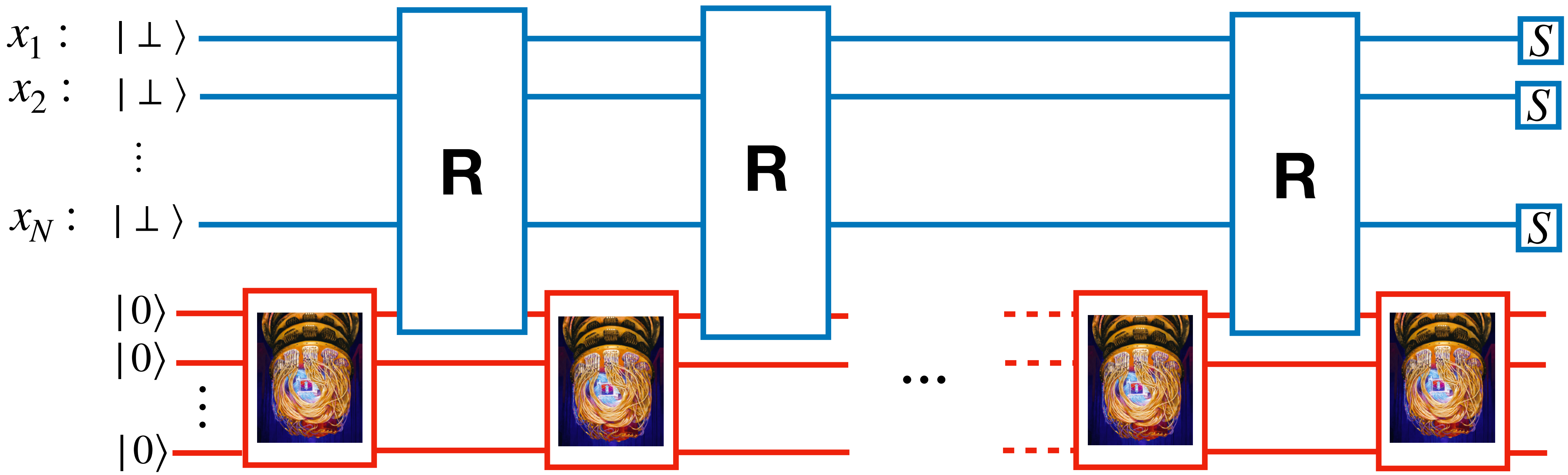



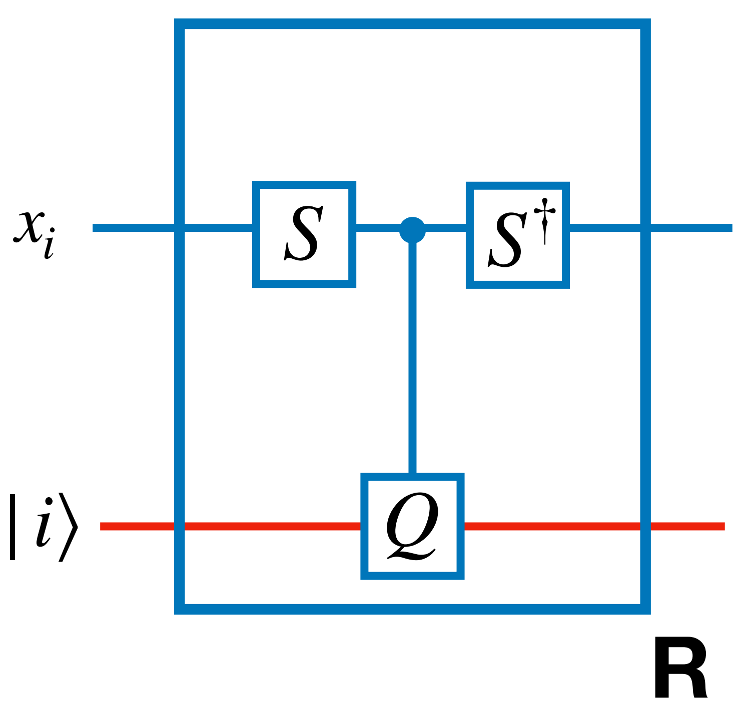
# Quantum Recording for K-Search

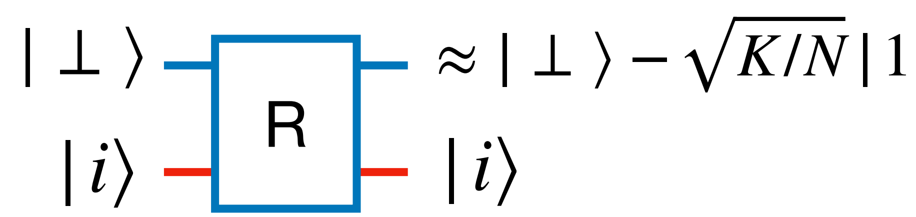


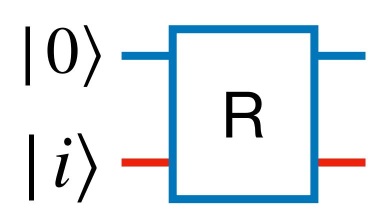


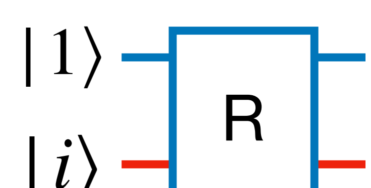

$$S|\perp\rangle = \sqrt{1 - K/N}\,|0\rangle + \sqrt{K/N}\,|1\rangle$$

## Recording Query Operator



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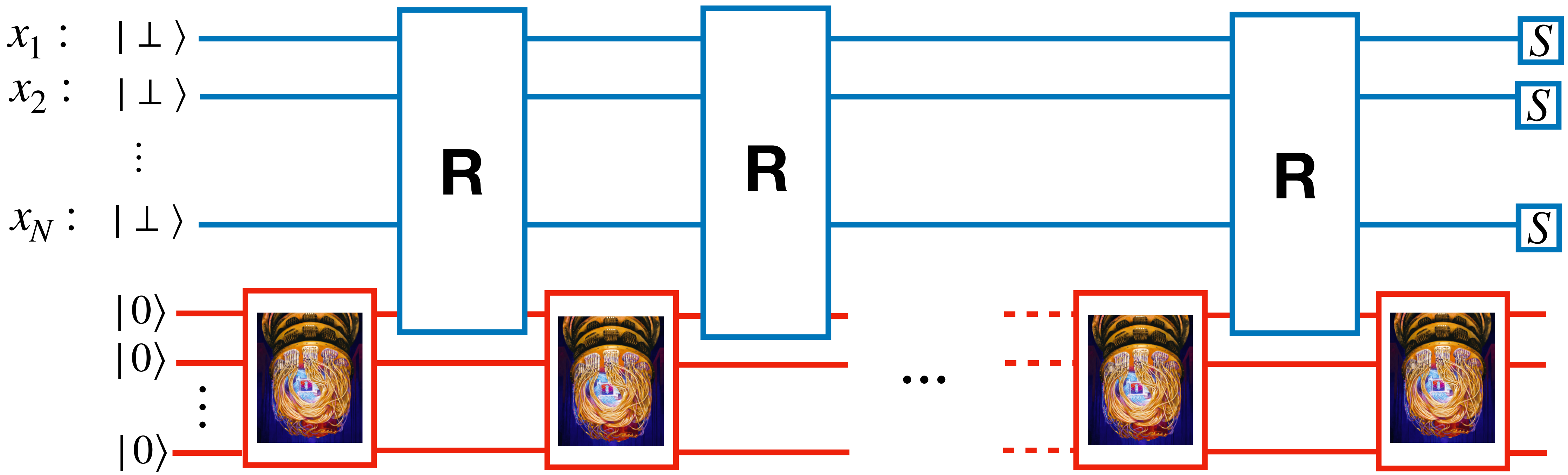


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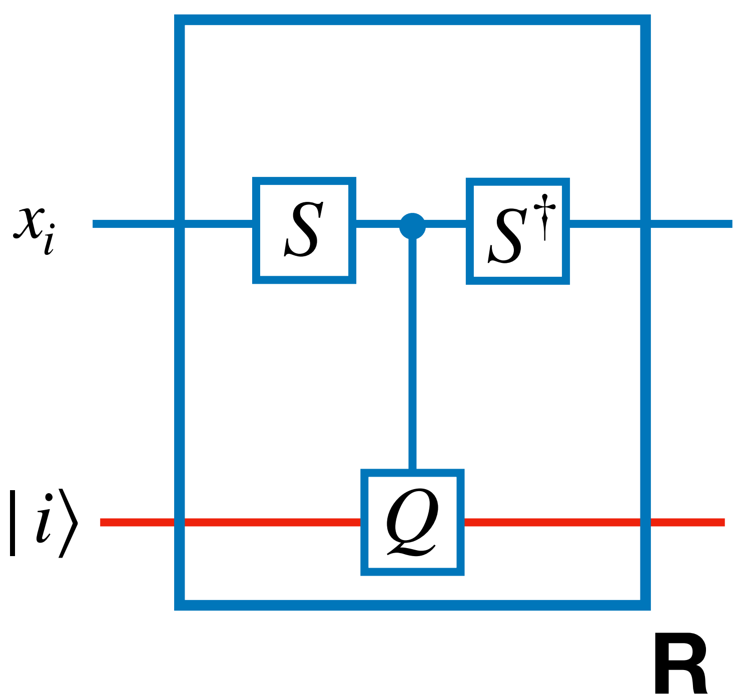


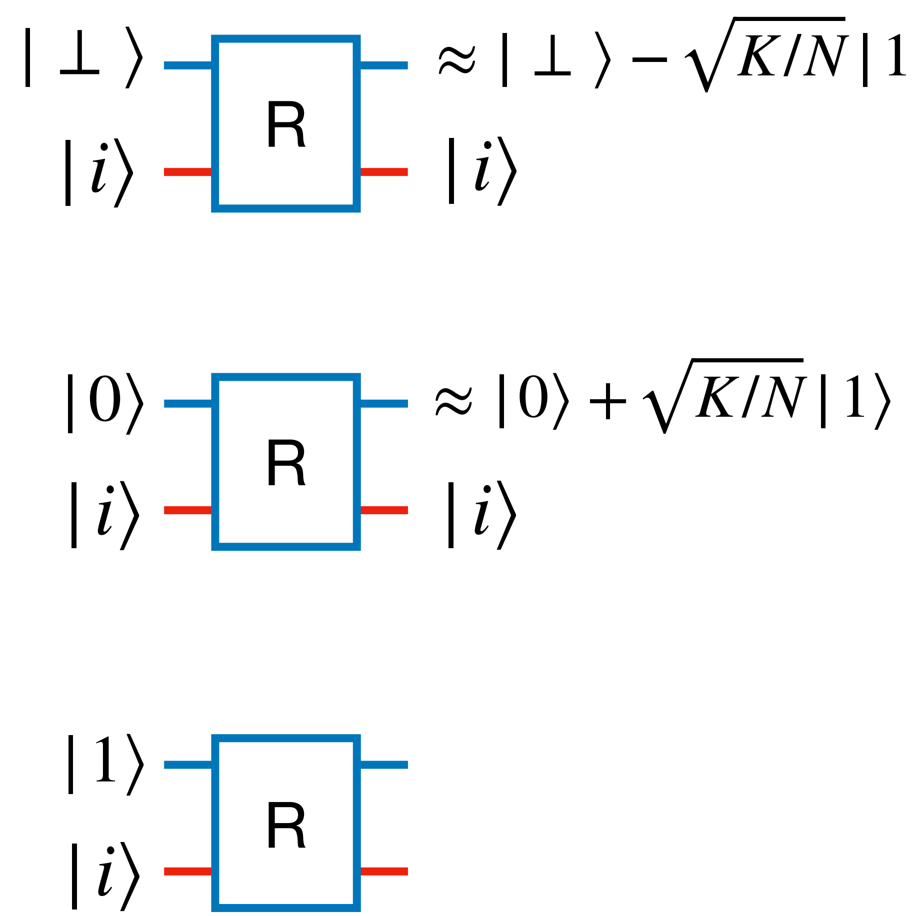

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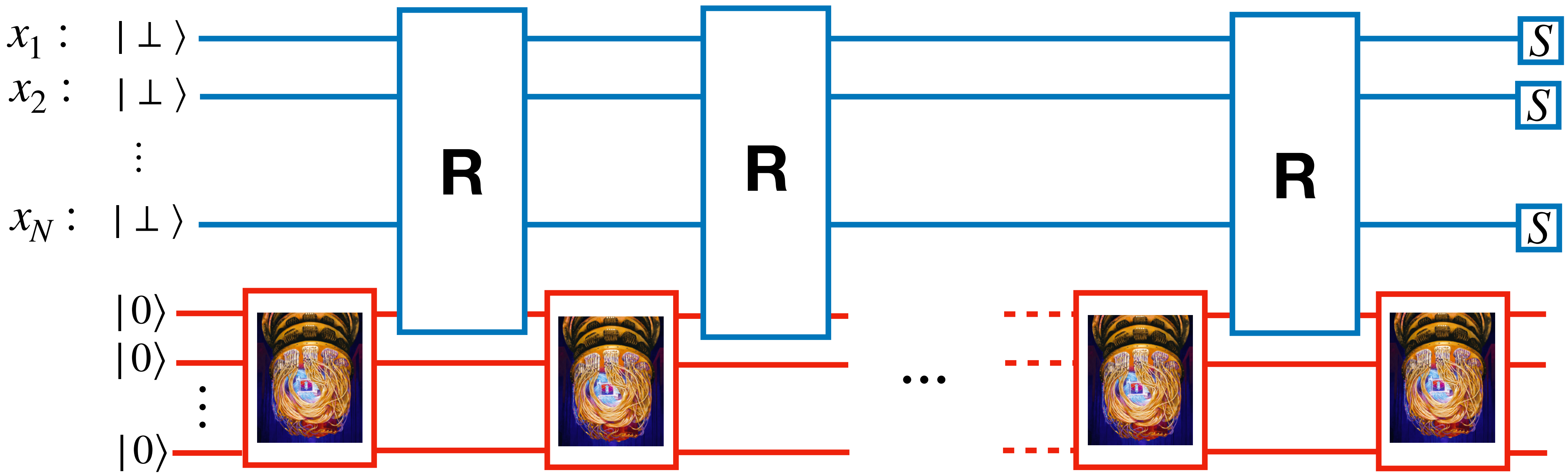


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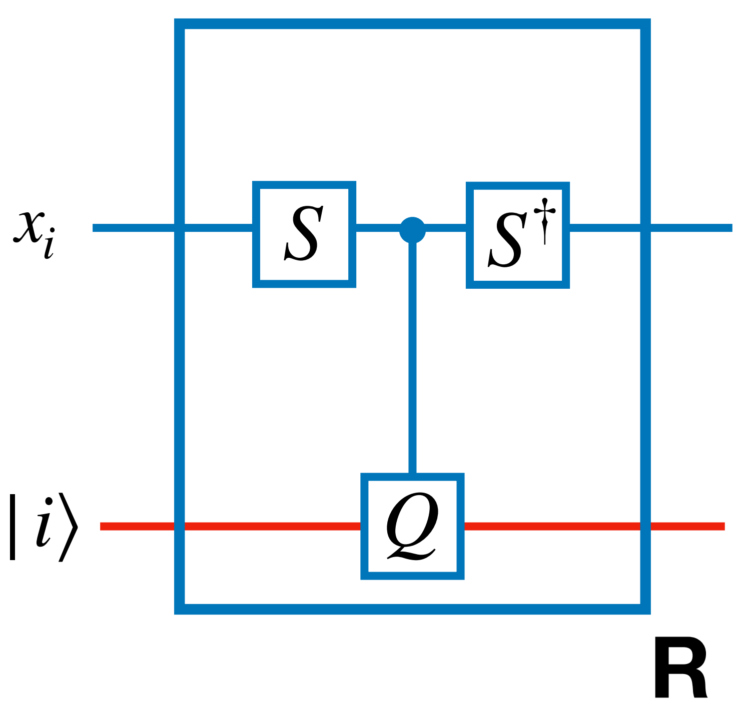


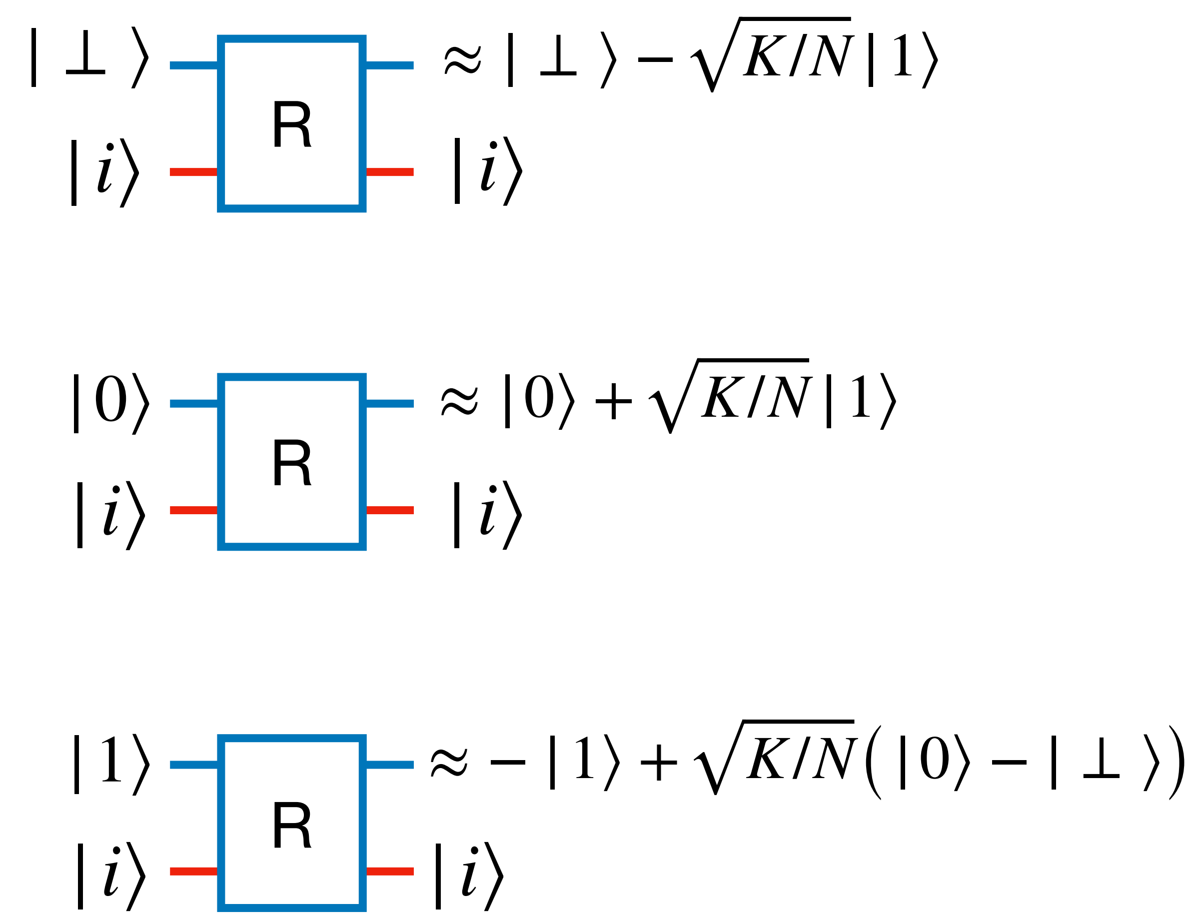

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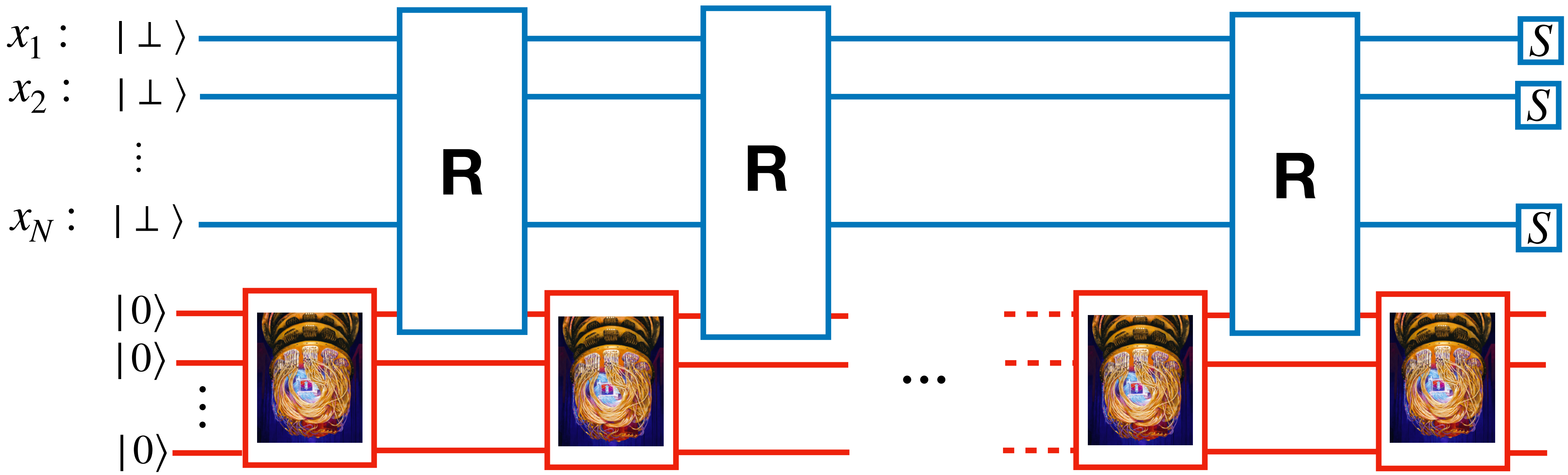


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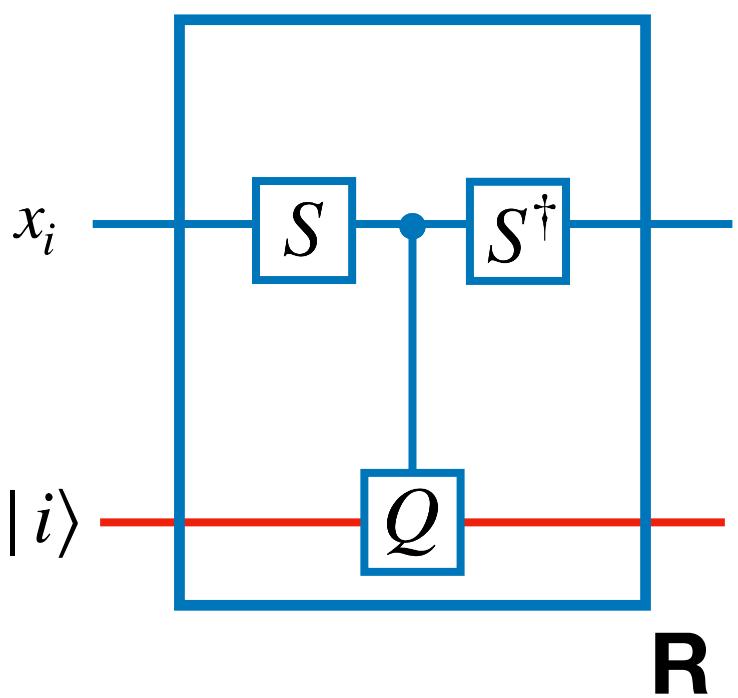


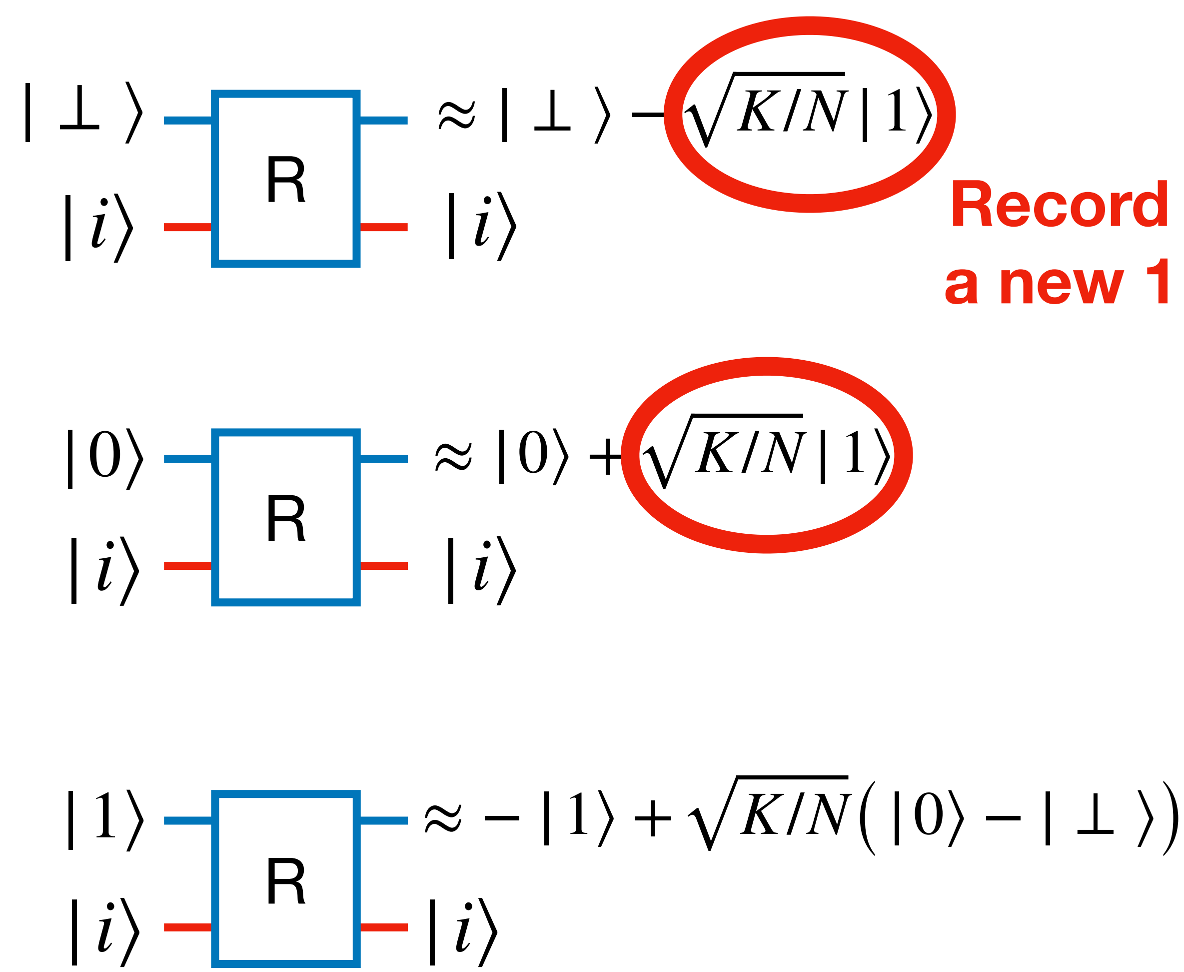

| | Classical Recording | Quantum Recording |
|---|---|---|
| **K-Search** | Probability to have recorded at least K/2 ones $\leq \binom{T}{K/2}\left(\frac{K}{N}\right)^{K/2}$ | |

| | Classical Recording | Quantum Recording |
| --- | --- | --- |
| **K-Search** | **Probability** to have recorded at least K/2 ones $\leq \binom{T}{K/2}\left(\frac{K}{N}\right)^{K/2}$ | **Amplitude** of the states that have recorded at least K/2 ones $\leq \binom{T}{K/2}\left(4\sqrt{\frac{K}{N}}\right)^{K/2}$ |

# Quantum Recording



| | Classical Recording | Quantum Recording |
|---|---|---|
| **K-Search** | **Probability** to have recorded at least K/2 ones $\leq \binom{T}{K/2}\left(\frac{K}{N}\right)^{K/2}$ | **Amplitude** of the states that have recorded at least K/2 ones $\leq \binom{T}{K/2}\left(4\sqrt{\frac{K}{N}}\right)^{K/2}$ |
| **K-Collision Pairs** | **Probability** to have recorded at least K/2 (disjoint) collisions $\leq \binom{T}{K/2}\left(\frac{T}{N}\right)^{K/2}$ | |

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# Quantum Recording



| | Classical Recording | Quantum Recording |
|---|---|---|
| **K-Search** | **Probability** to have recorded at least K/2 ones<br><br>$\leq 2^{-\Omega(K)}$ when $T \leq O(N)$ | **Amplitude** of the states that have recorded at least K/2 ones<br><br>$\leq 2^{-\Omega(K)}$ when $T \leq O\left(\sqrt{NK}\right)$ |
| **K-Collision Pairs** | **Probability** to have recorded at least K/2 (disjoint) collisions<br><br>$\leq 2^{-\Omega(K)}$ when $T \leq O\left(\sqrt{NK}\right)$ | **Amplitude** of the states that have recorded at least K/2 (disjoint) collisions<br><br>$\leq 2^{-\Omega(K)}$ when $T \leq O\left(K^{2/3}N^{1/3}\right)$ |

# Conclusion

# Open Problems:

- Extend the quantum recording technique to **non-product** distributions?

  Example: uniform distribution over the symmetric group.

- Improve the tradeoff for finding $\tilde{\Theta}(N)$ Collision Pairs to $T^2S \geq \Omega(N^3)$,

  or find a quantum algorithm with $T^3S \leq O(N^4)$?

- New lower bounds by recording queries? **Triangles Finding**?

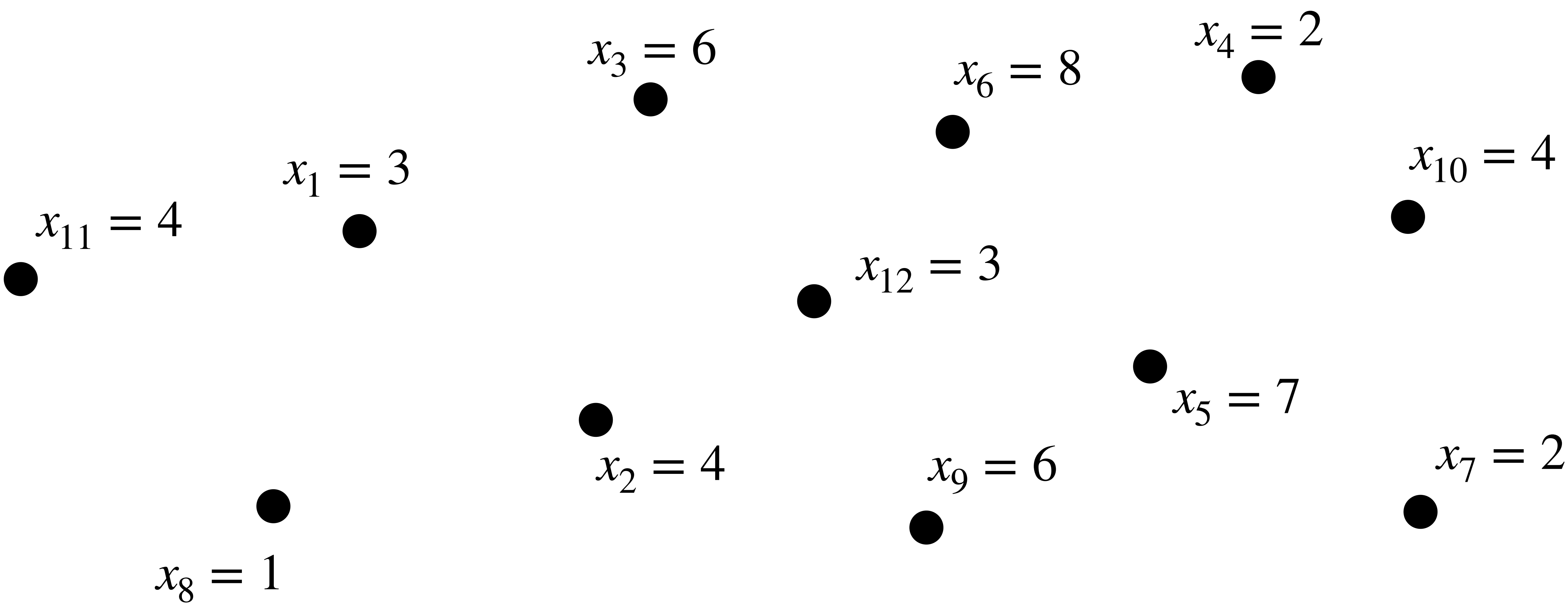
**arXiv: 2002.08944**

# Supplementary slides

# Reducing Θ̃(N)-Collision Pairs to Element Distinctness



How to find **Θ̃(N) Collision Pairs** by using an algorithm for **Element Distinctness**



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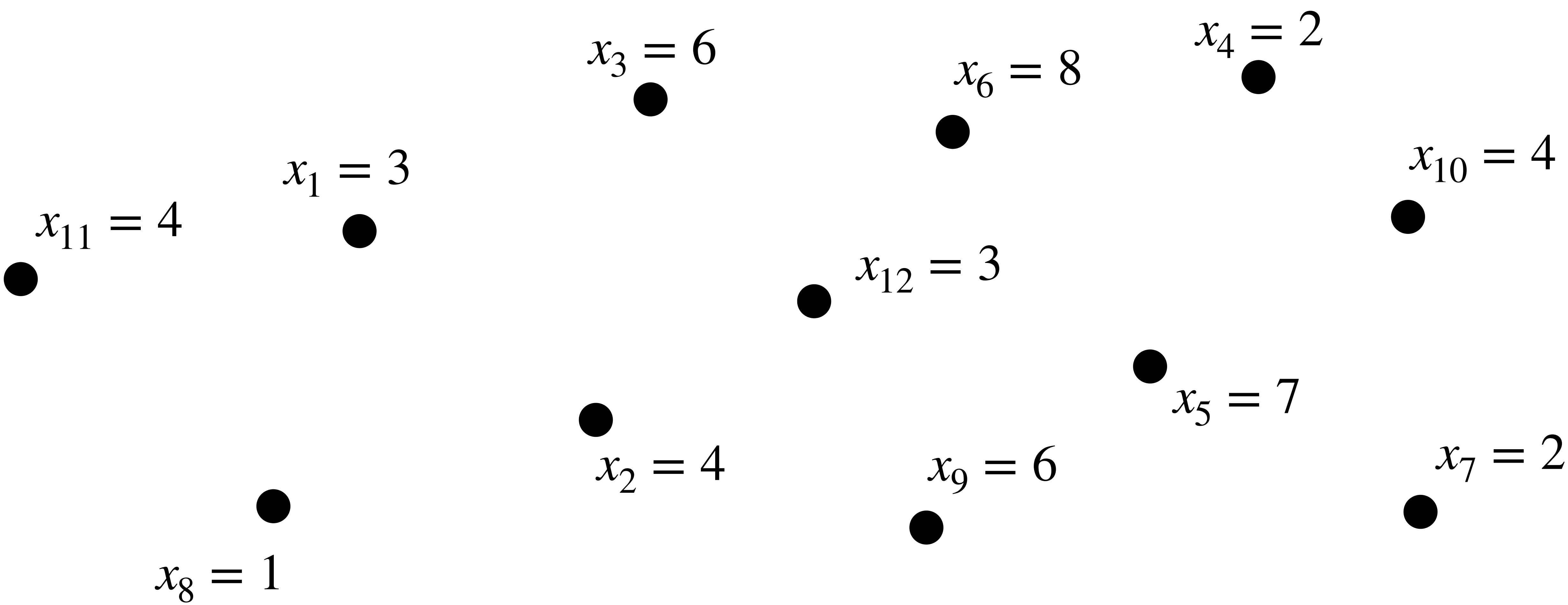


**Repeat O(N) times:** sample $\sqrt{N}$ elements and find a collision among them with ED.

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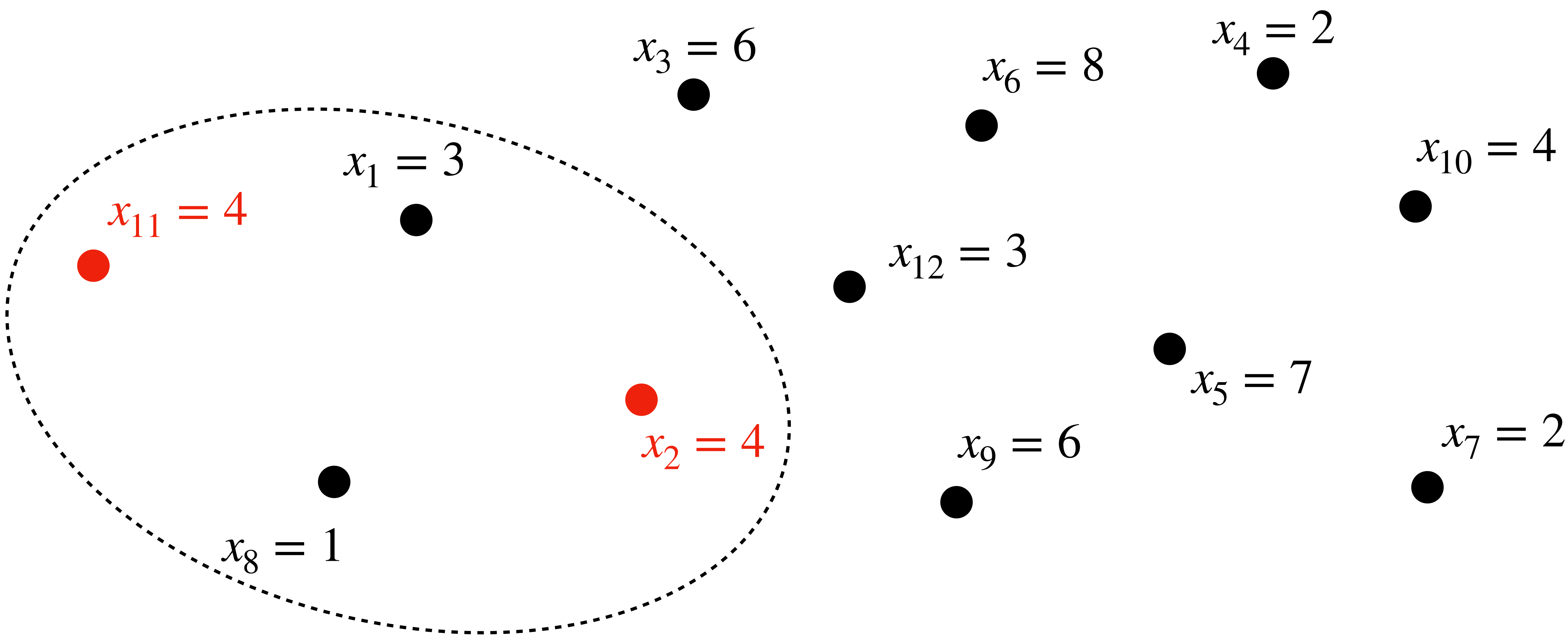


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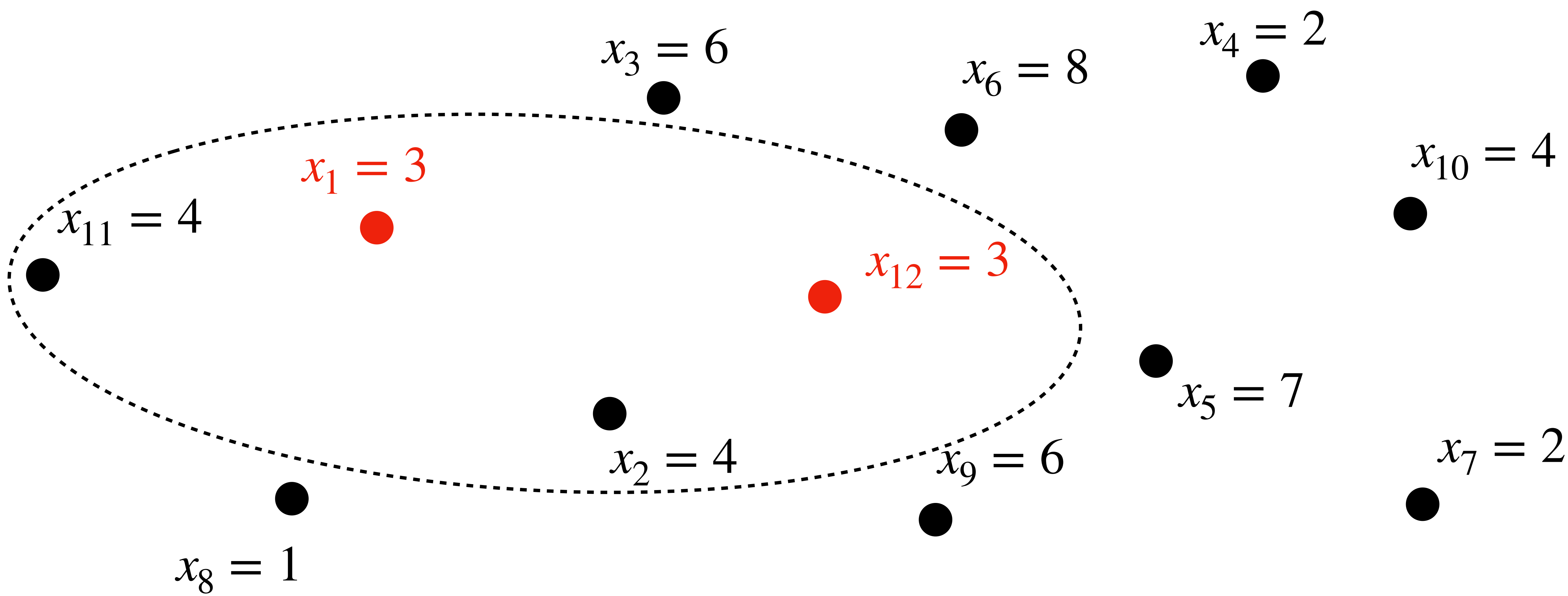


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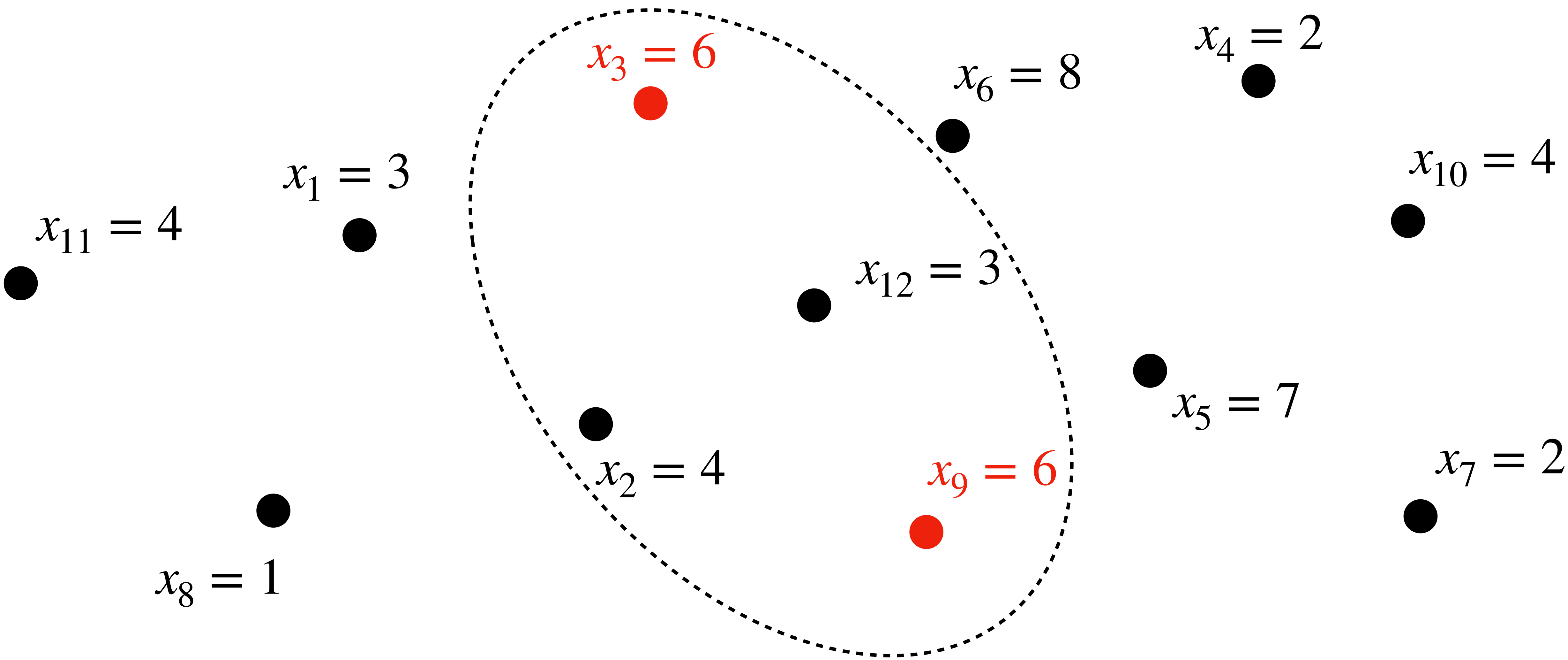


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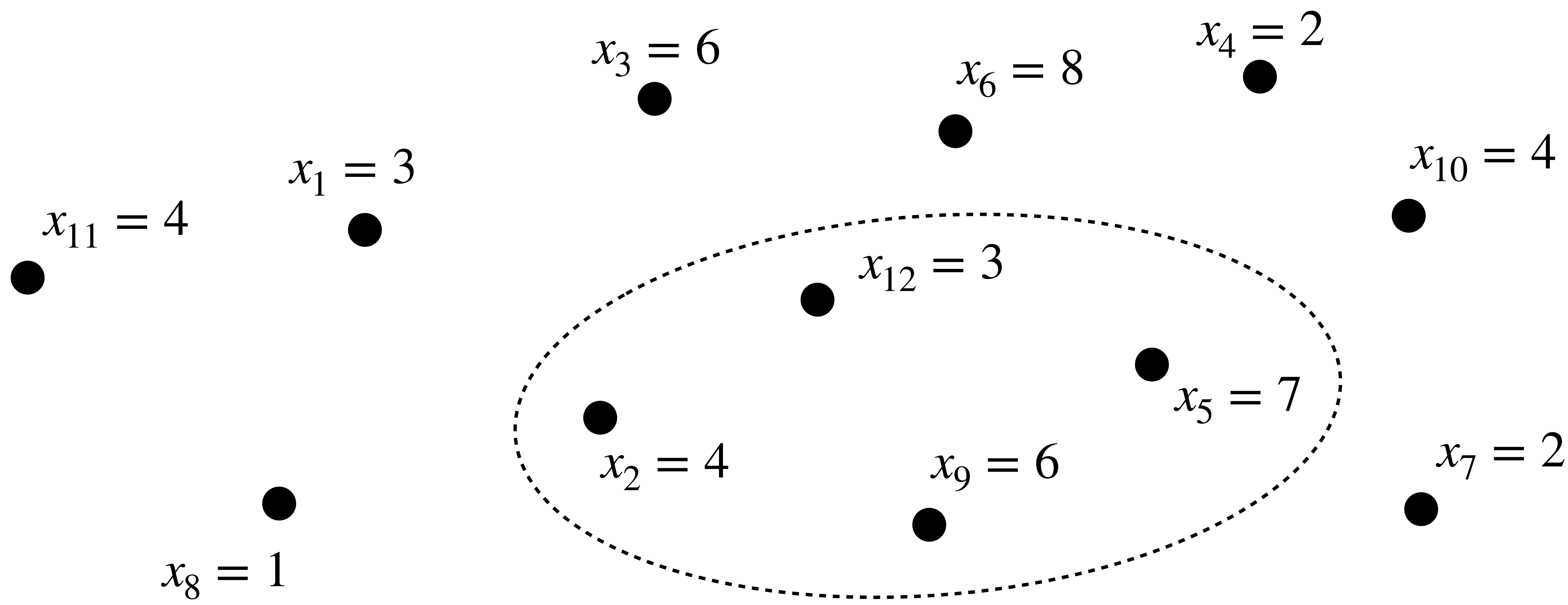


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→ Sometimes, there is no collision to find.

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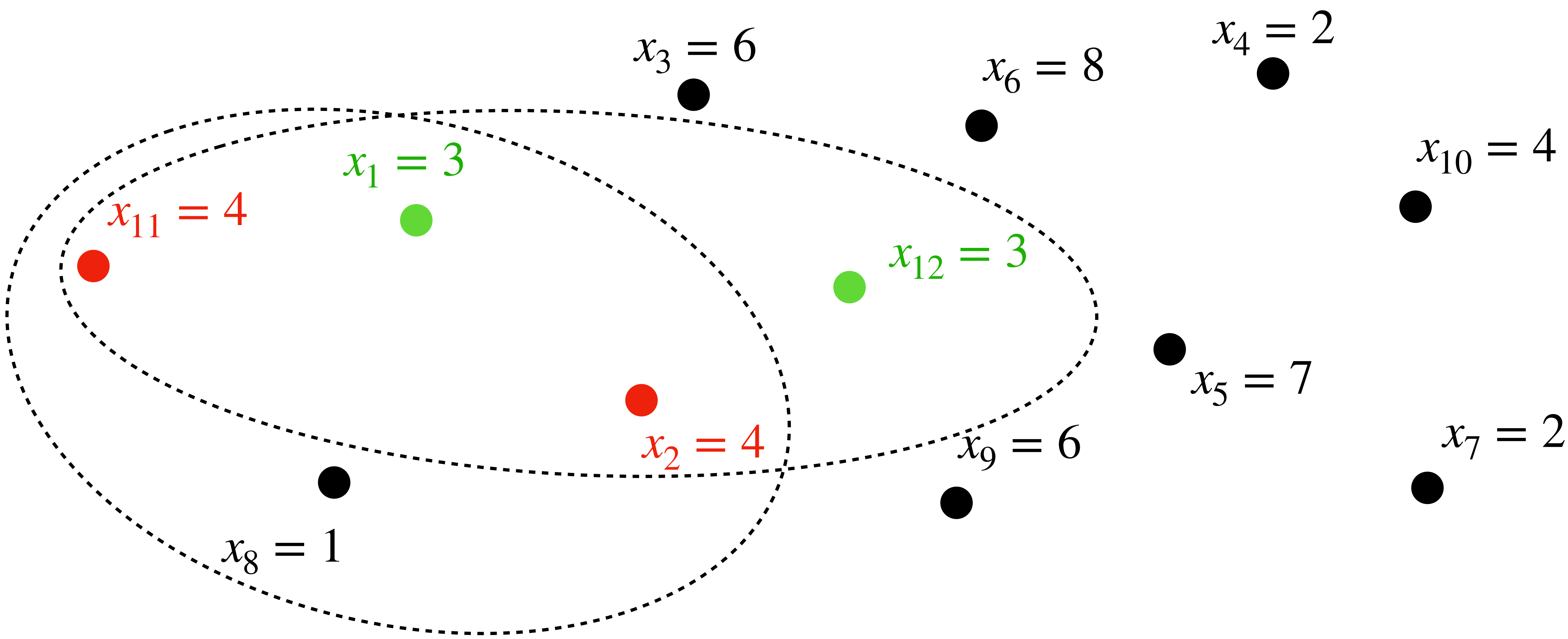


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→ Sometimes, there is no collision to find.

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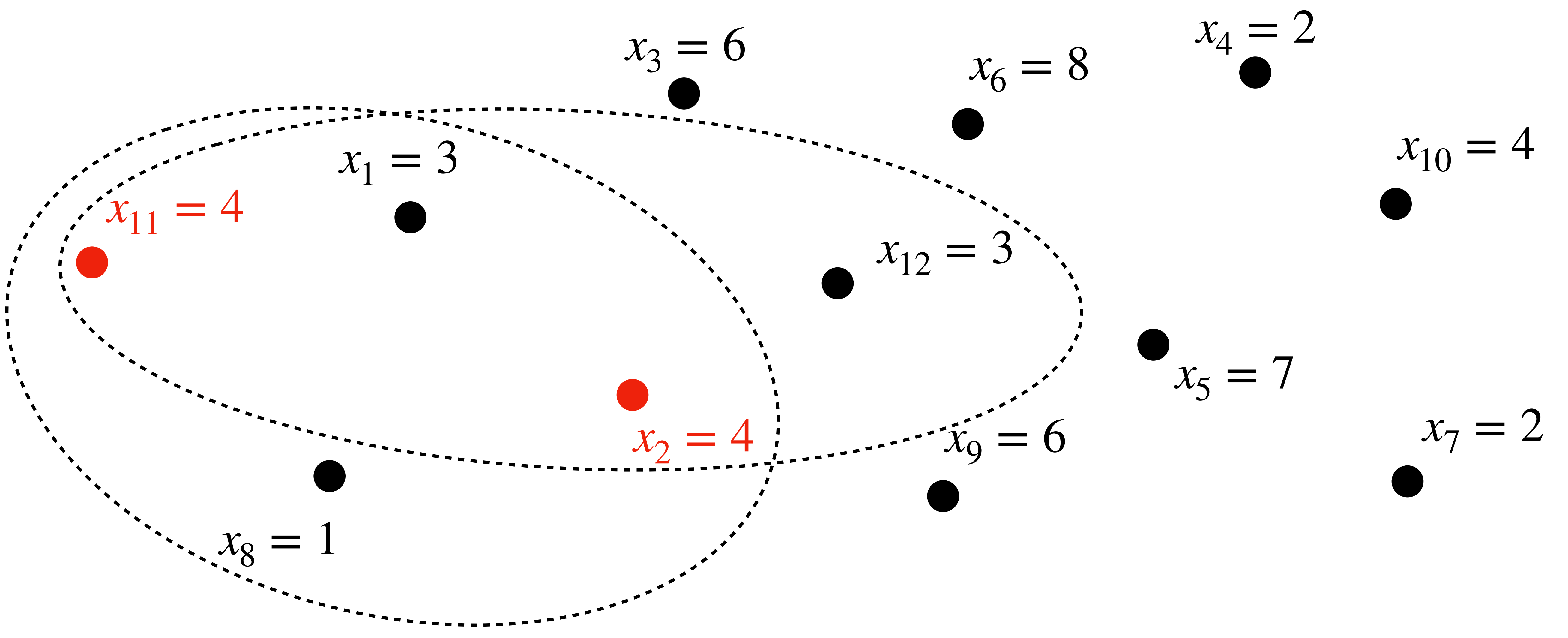
How to find **$\tilde{\Theta}(N)$ Collision Pairs** by using an algorithm for **Element Distinctness**

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→ Sometimes, there is no collision to find.

→ We cannot control which collision ED is going to output.

→ We can output the same collision many times (but it only counts as one collision).

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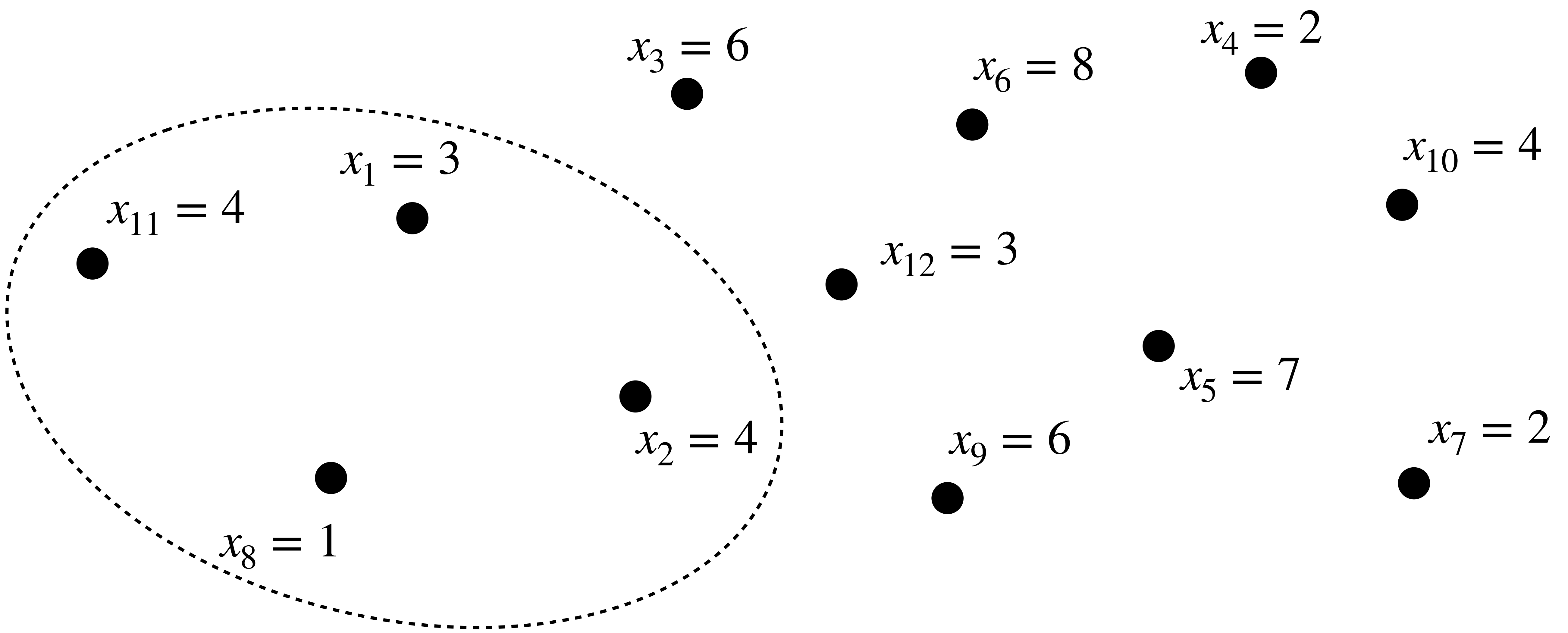


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→ Sometimes, there is no collision to find.

→ We cannot control which collision ED is going to output.

→ We can output the same collision many times (but it only counts as one collision).

→ We need to store the $\sqrt{N}$ sampled indices ⇒ **4-wise independent sampling**

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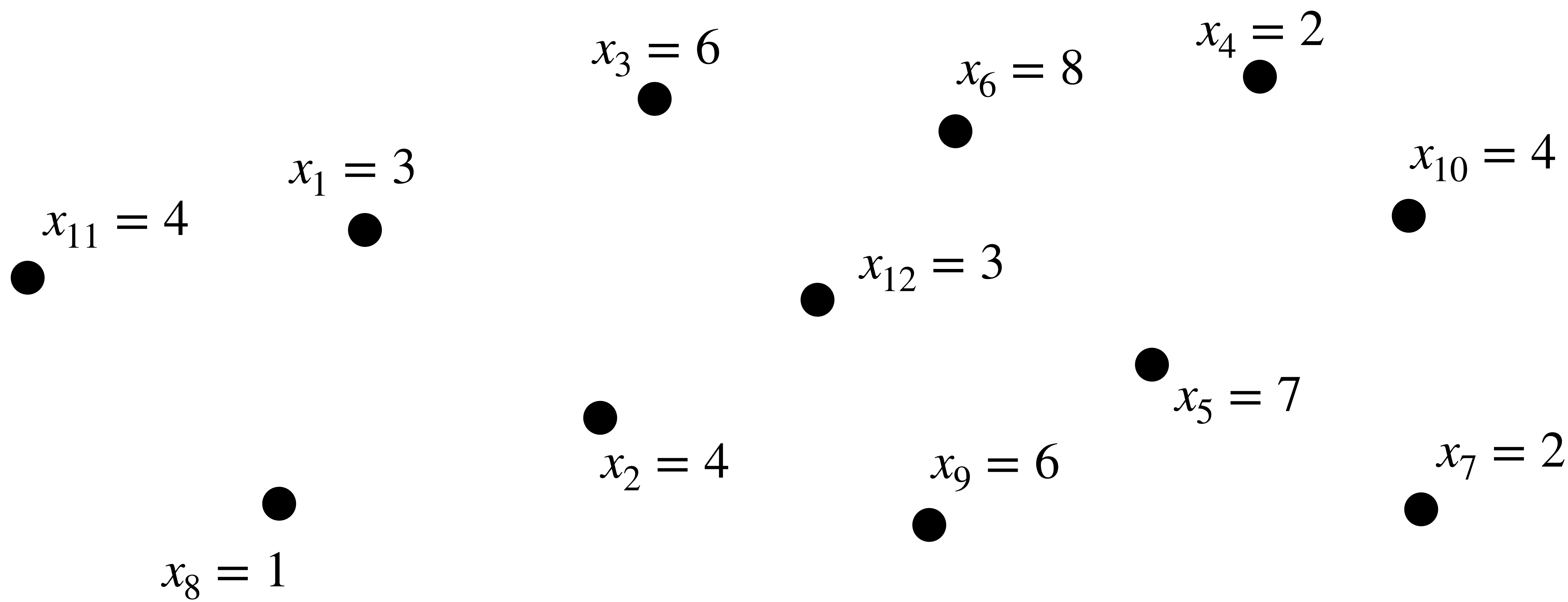


**(T,S)**-Algorithm for **Element Distinctness** on inputs of size $\sqrt{N}$ $\Longrightarrow$ **(NT,S)**-Algorithm for finding **Θ̃(N) Collision Pairs** on inputs of size $N$

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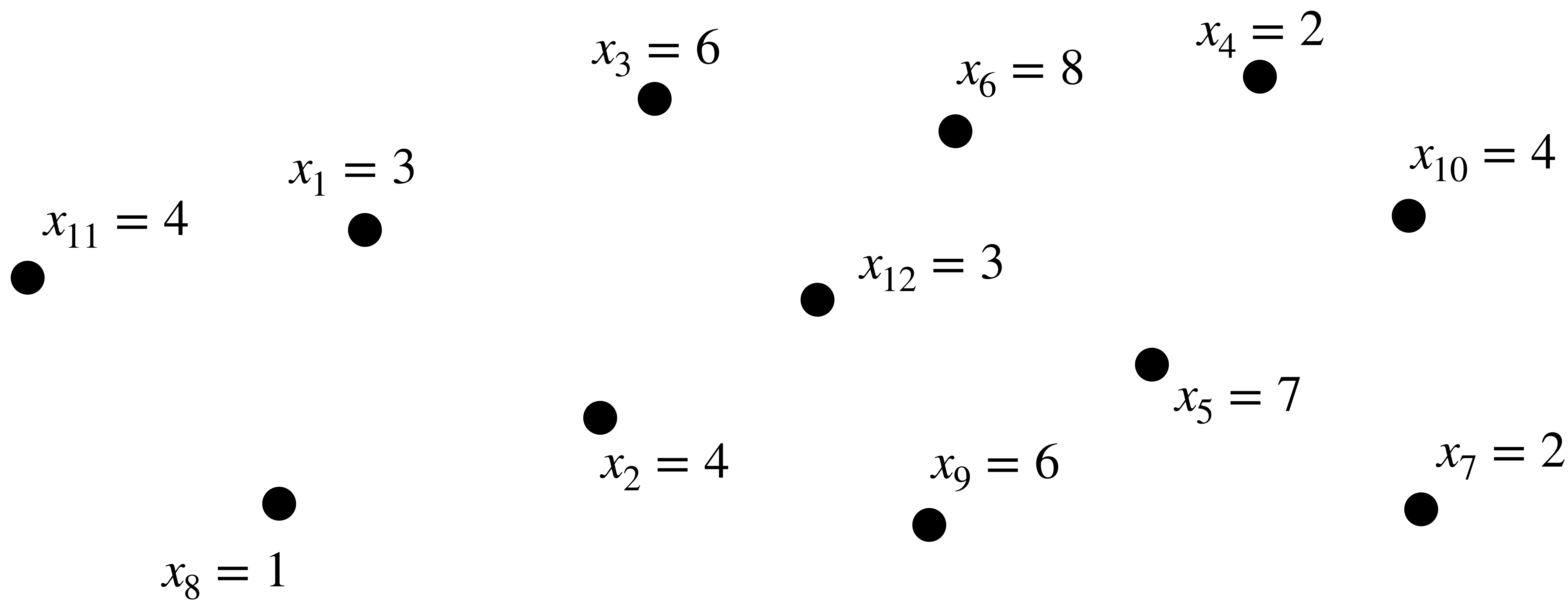


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$$(NT)^2 S \geq \tilde{\Omega}\left(N^3\right)$$

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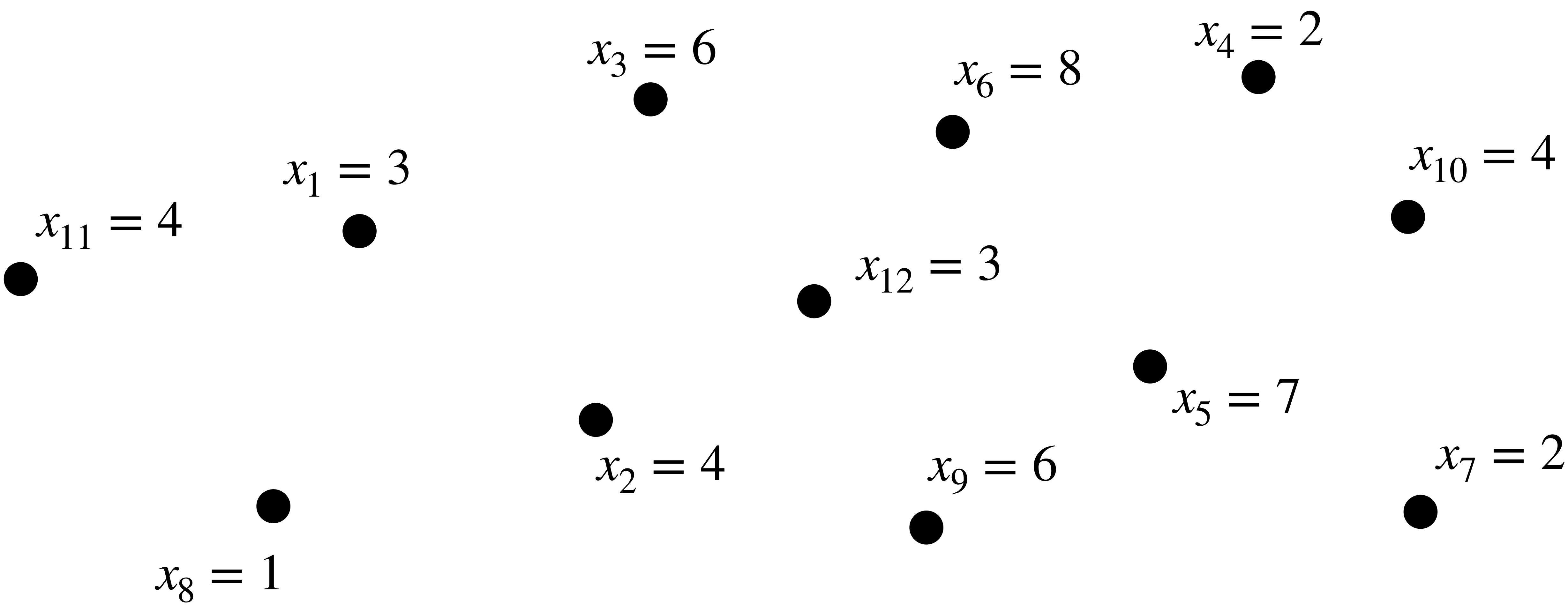


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$T^2 S \geq \tilde{\Omega}\left(\sqrt{N}^2\right)$ $\Longleftarrow$ $(NT)^2 S \geq \tilde{\Omega}\left(N^3\right)$