Classical and quantum algorithms for variants of Subset-Sum via dynamic programming

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joint work with

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The problems

Subset-Sum

Input: Multiset $\{a_1, \ldots, a_n\}$ and target m

Output: Subset $S \subseteq [n]$ such that $\sum_{i \in S} a_i = m$

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Pigeonhole Equal-Sums

Assumption: $s = 0$ and $\sum_{i=1}^{n} a_i < 2^n - 1$

Results

Prior work	Classical	Quantum
SUBSET-SUM [HS'74,BJLM'13]	$\widetilde{O}(2^{n/2})$	$\widetilde{O}(2^{n/3})$
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SHIFTED-SUMS	$O(2^{0.773n})$	$O(2^{0.504n})$
PIGEONHOLE EQUAL-SUMS	$\widetilde{O}(2^{n/2})$	$\widetilde{O}(2^{2n/5})$

Main idea for Shifted-Sums

Representation technique approach:

– Standard in average-case analysis of Subset-Sum [HJ'10]

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Good choice for p and k:

– the bins are small to keep the cost of search low.

– the bins contain a solution with large probability.

Number of collision values

Collision values:

$$V = \left\{ v \in \mathbb{N} : \exists S_1 \neq S_2, v = \sum_{i \in S_1} a_i = \sum_{i \in S_2} a_i + s \right\}$$

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Lemma: At least $|V| \geq 2^{n-\gamma}$ distinct collision values.

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Difficulty: each step requires one (quantum) query to $T_{p,k}$

$\Rightarrow$ for some indexing of $T_{p,k} = \{S_1, \ldots, S_{|T_{p,k}|}\}$ we need to implement the oracle

$$O_{T_{p,k}} : \ell \mapsto S_\ell, \quad \text{for} \;\; 1 \leq \ell \leq |T_{p,k}|$$

Dynamic programming data structure

Compute the cardinality table

$$t_p[i,j] = |\{S \subseteq \{1,\ldots,i\} : \sum_{i \in S} a_i \equiv j \pmod{p}\}|$$

by dynamic programming in time $O(np)$.

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Theorem: Given $1 \leq \ell \leq t_{p,k}$ and random access to the elements of the table t_p, the ℓ-th set in $T_{p,k}$ can be computed in time $O(n)$.

Computing the oracle

Input: Table t_p, integers $0 \leq k \leq p-1$ and $1 \leq \ell \leq t_{p,k}$
Output: ℓ-th set $S \in T_{p,k}$ for $\prec$.

❶ $j = k, S = \varnothing$

❷ **for** $i = n, \ldots, 1$ **do**

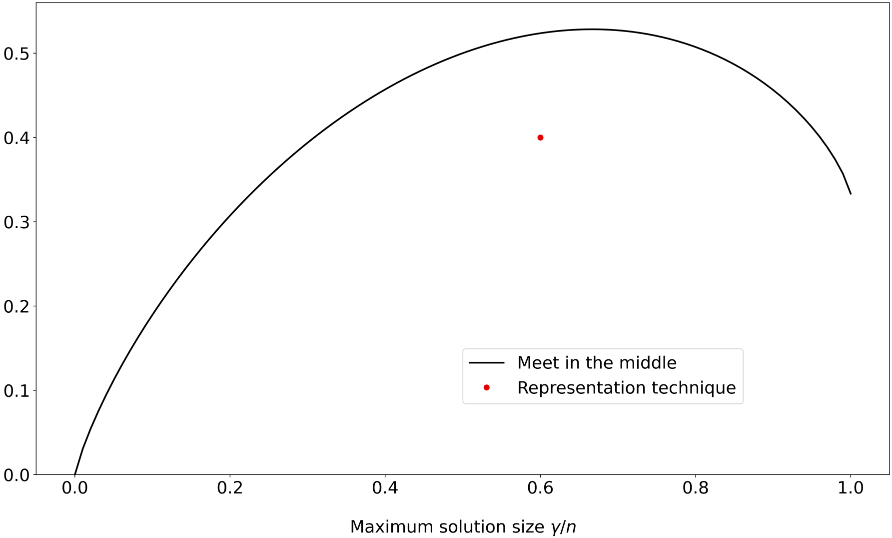
❸ **if** $\ell \leq t_p[i-1, j]$ **then** Do nothing

❹ **else**

❺ $S = S \cup \{i\}, \quad \ell = \ell - t_p[i-1, j], \quad j = j - a_i \mod p$

❻ Return S

Running time exponent



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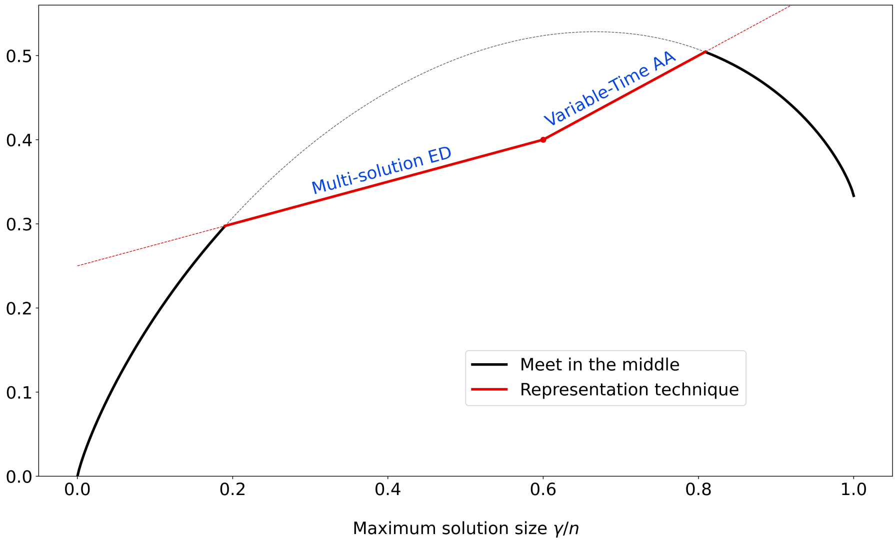

Open problem

PIGEONHOLE MODULAR EQUAL-SUMS
Input: Set $\{a_1, \ldots, a_n\}$ and a modulus q s.t. $q \leq 2^n - 1$
Output: Subsets $S_1 \neq S_2 \subseteq [n]$ s.t. $\sum_{i \in S_1} a_i \equiv_q \sum_{i \in S_2} a_i$

Generalizes PIGEONHOLE EQUAL-SUMS

Theorem: Can be solved deterministically in time $\widetilde{O}(2^{n/2})$

Question: Can be solved quantumly in time $\widetilde{O}(2^{2n/5})$?