Quantum and Classical Algorithms for Approximate Submodular Function Minimization

Yassine Hamoudi, Patrick Rebentrost,

Ansis Rosmanis, Miklos Santha

arXiv: 1907.05378

1. Approximate Submodular Function Minimization

2. Quantum speed-up for Importance Sampling

1

Approximate Submodular Function Minimization

Submodular Function

A **submodular function** is a set function $F : 2^{[n]} \to \mathbb{R}$ satisfying the **diminishing returns property**:

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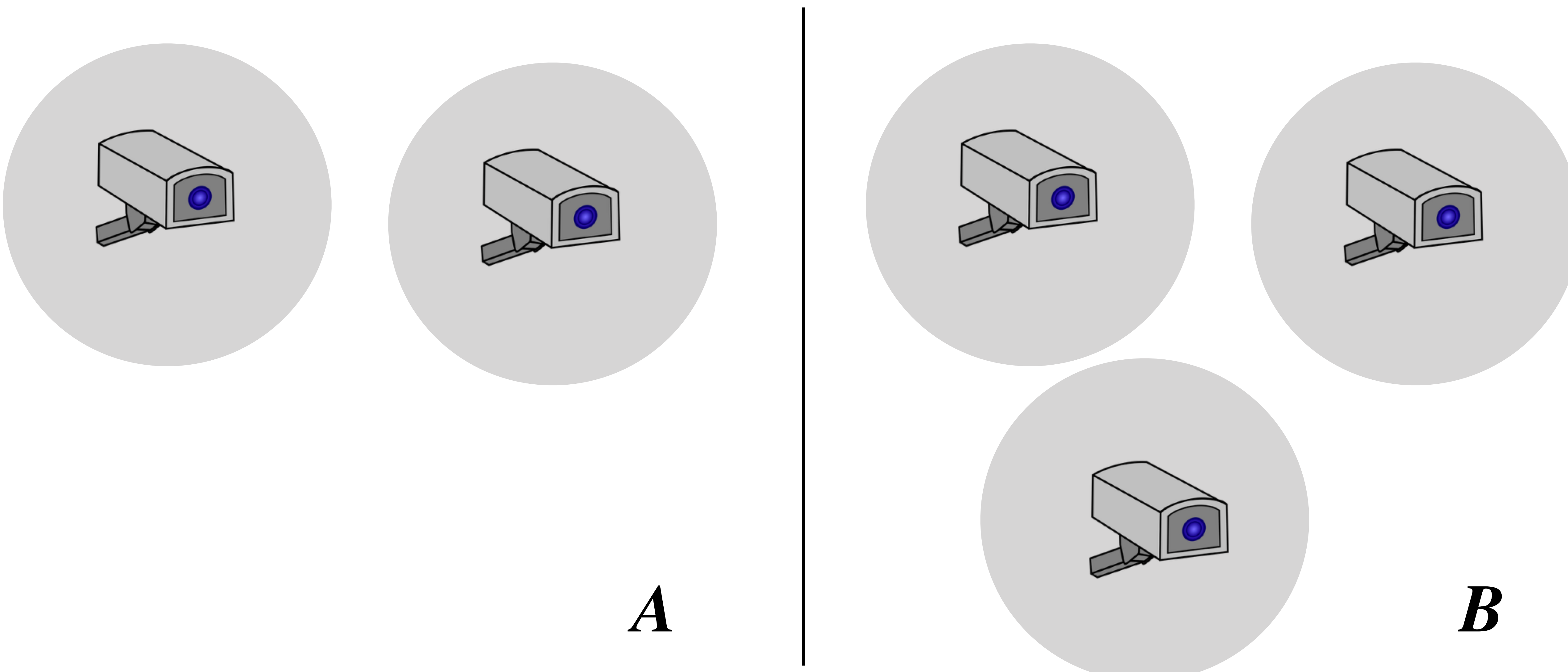

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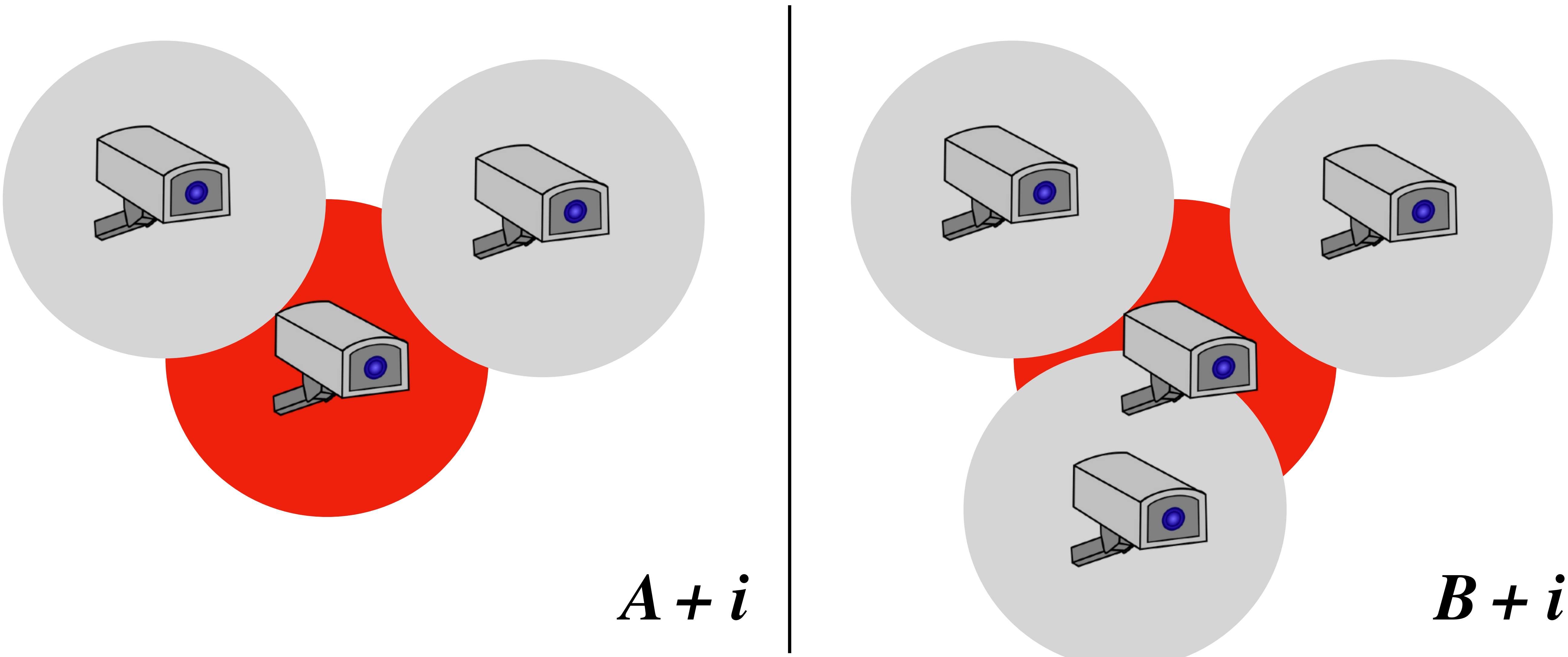

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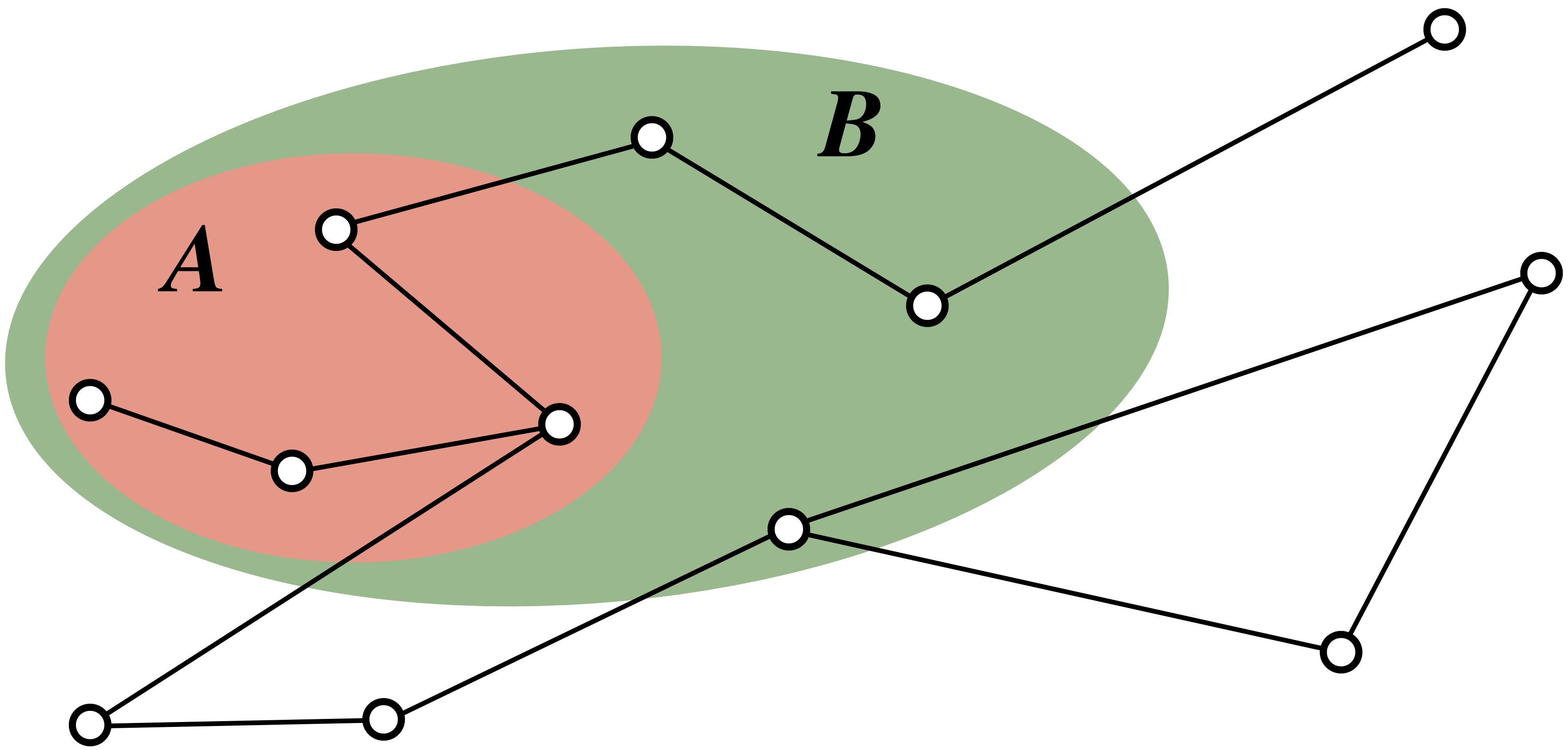

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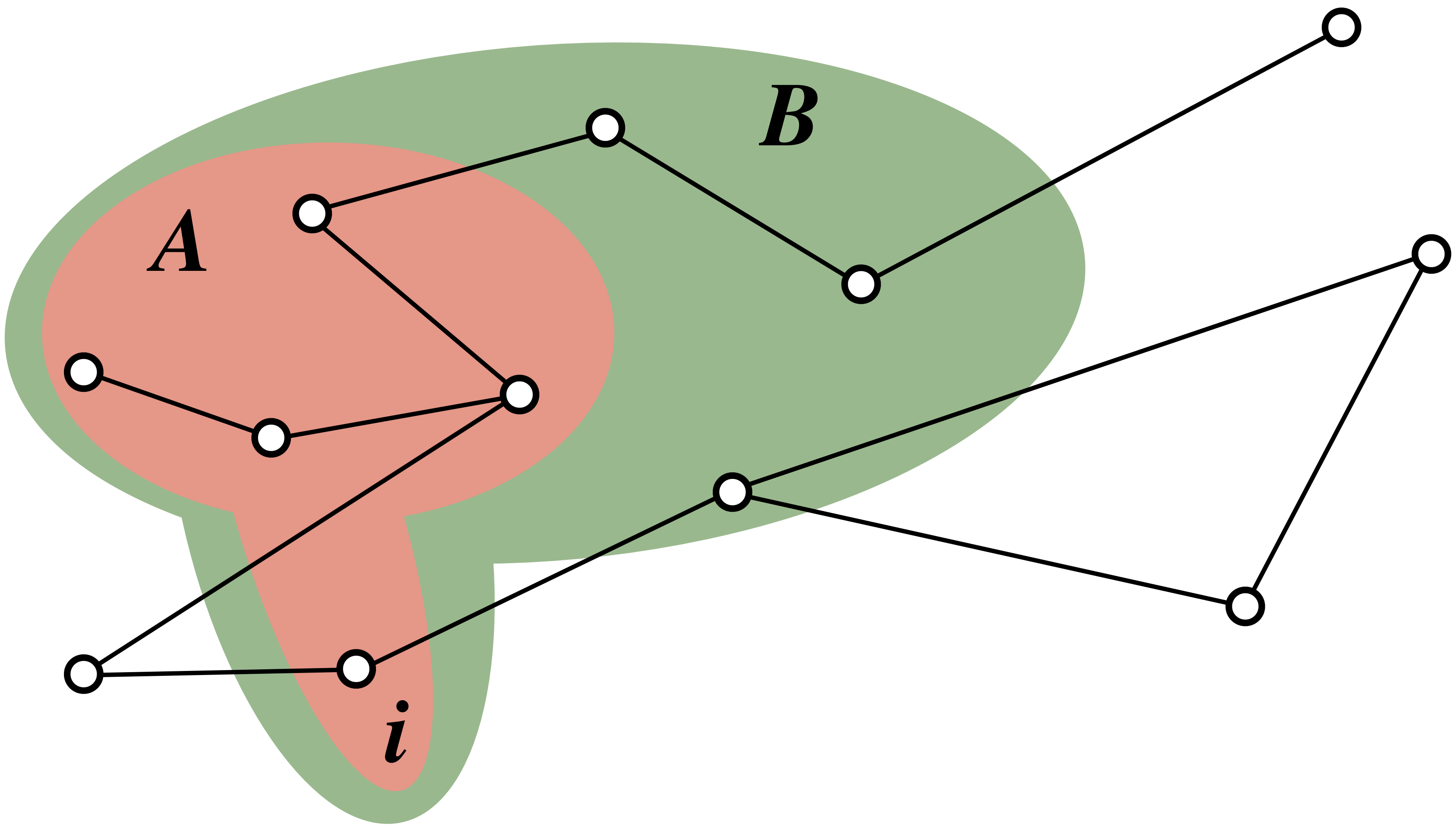

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Set function: $F : 2^{[n]} \to \mathbb{R}$

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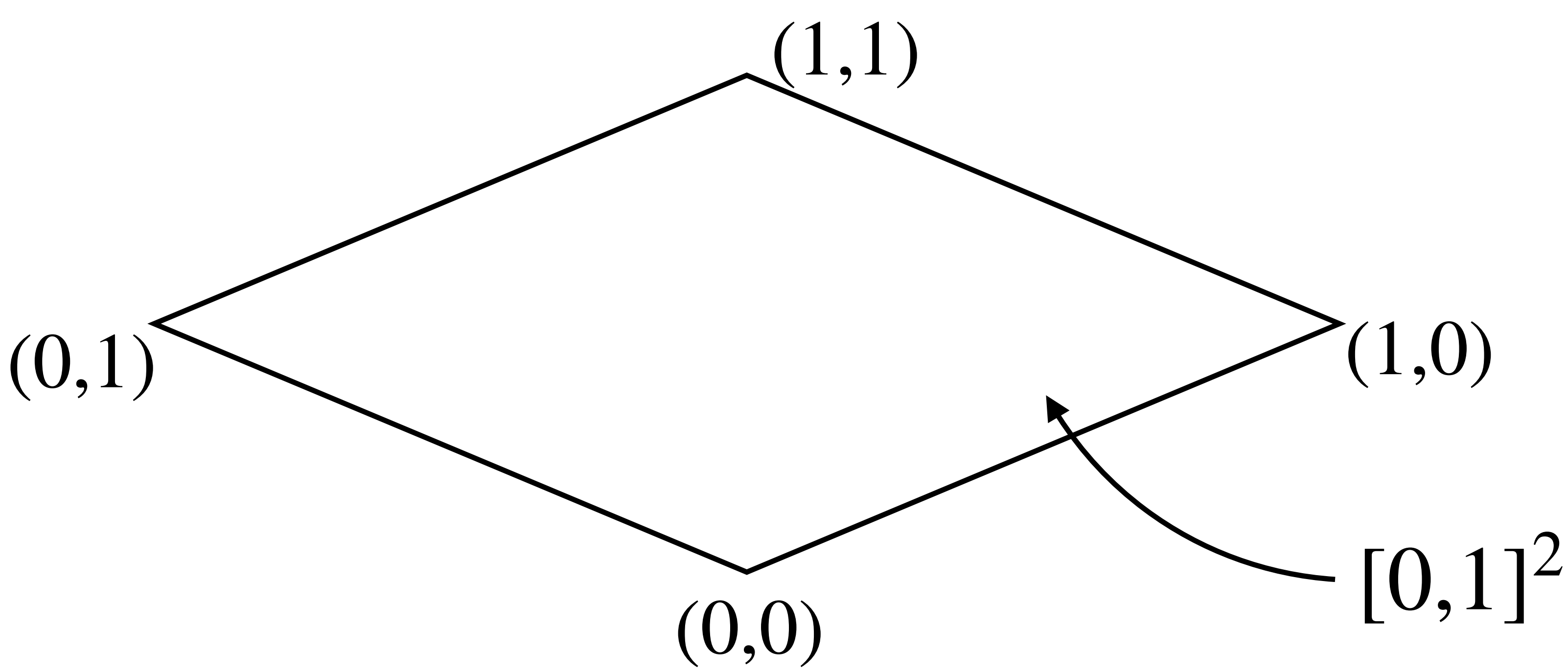

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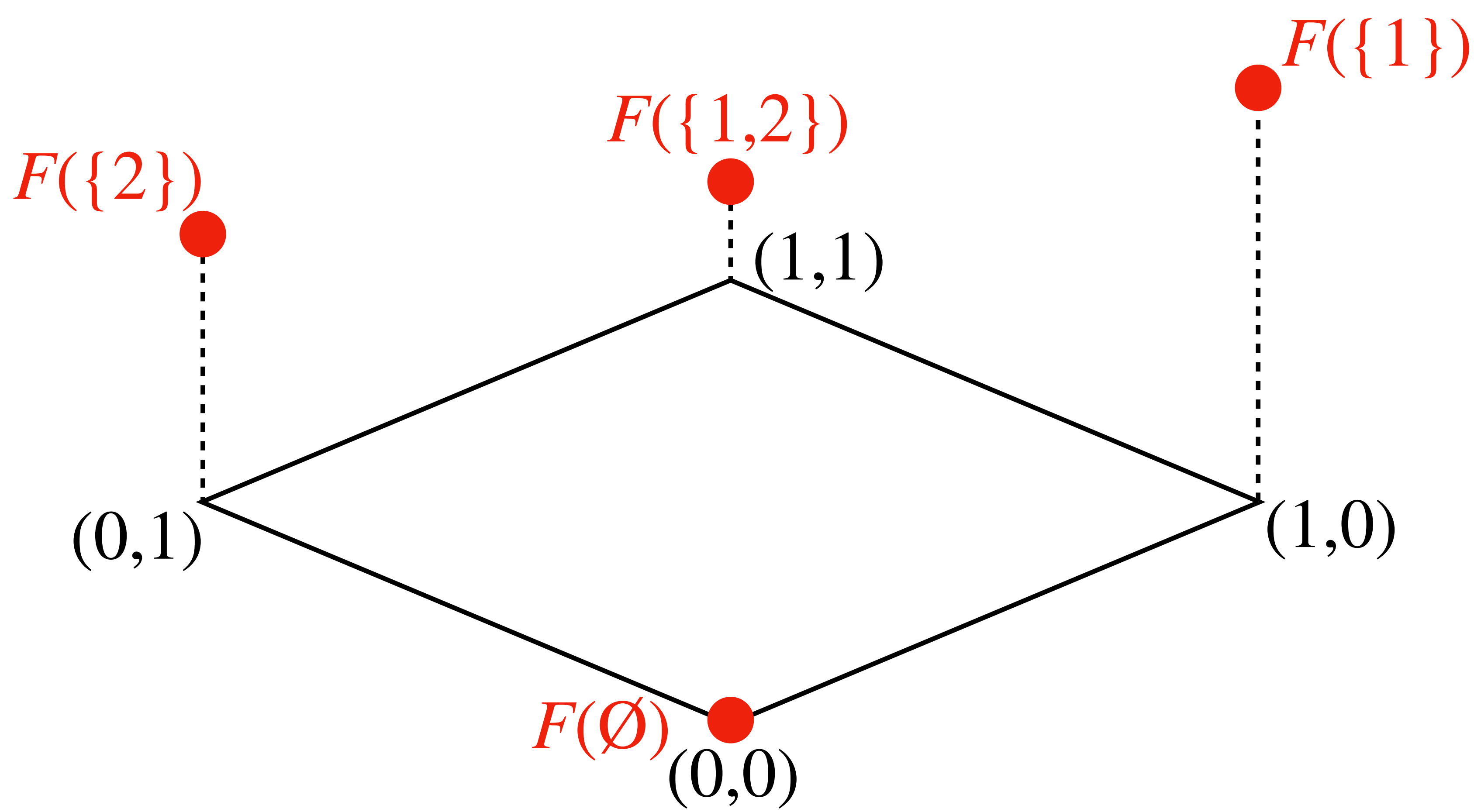

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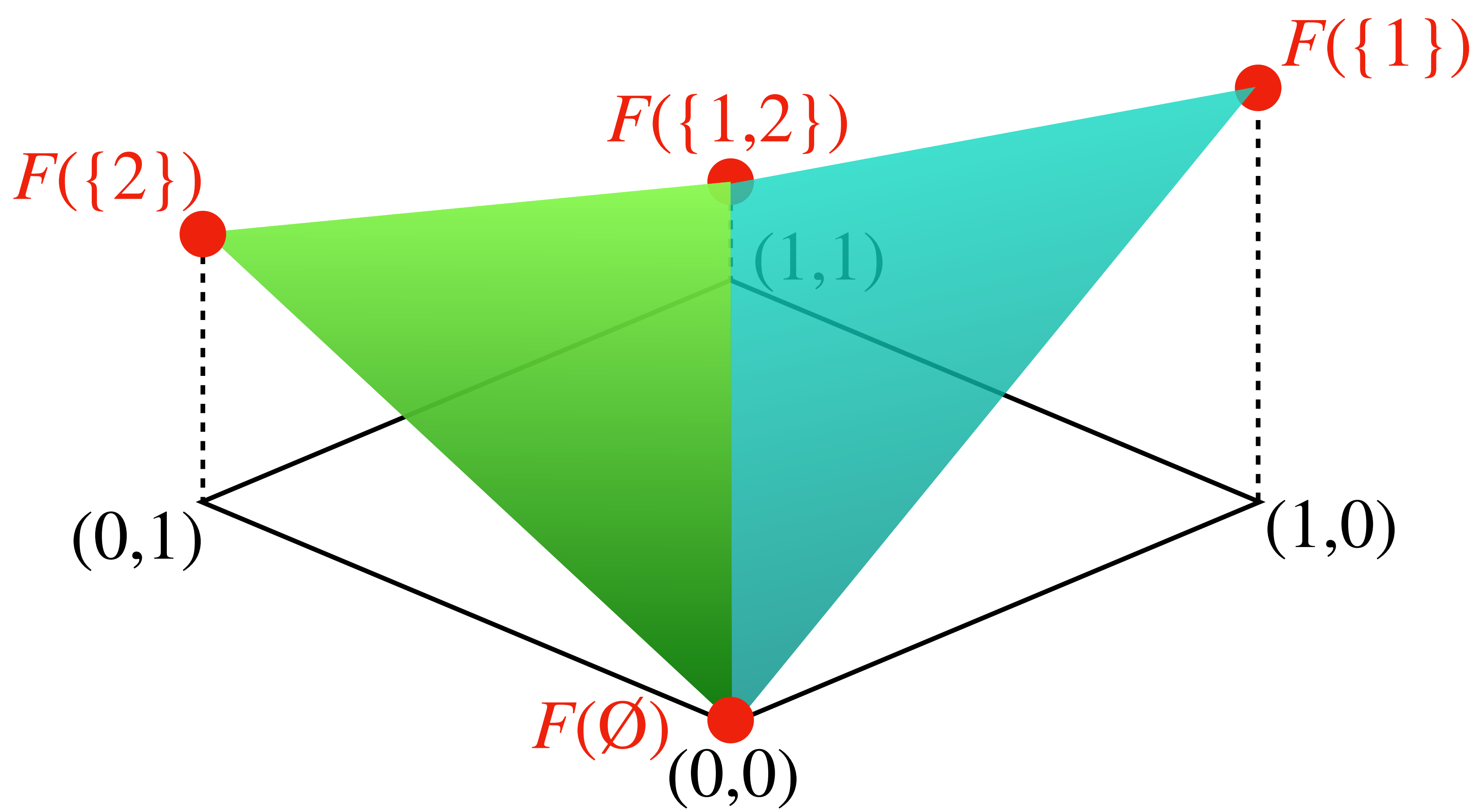

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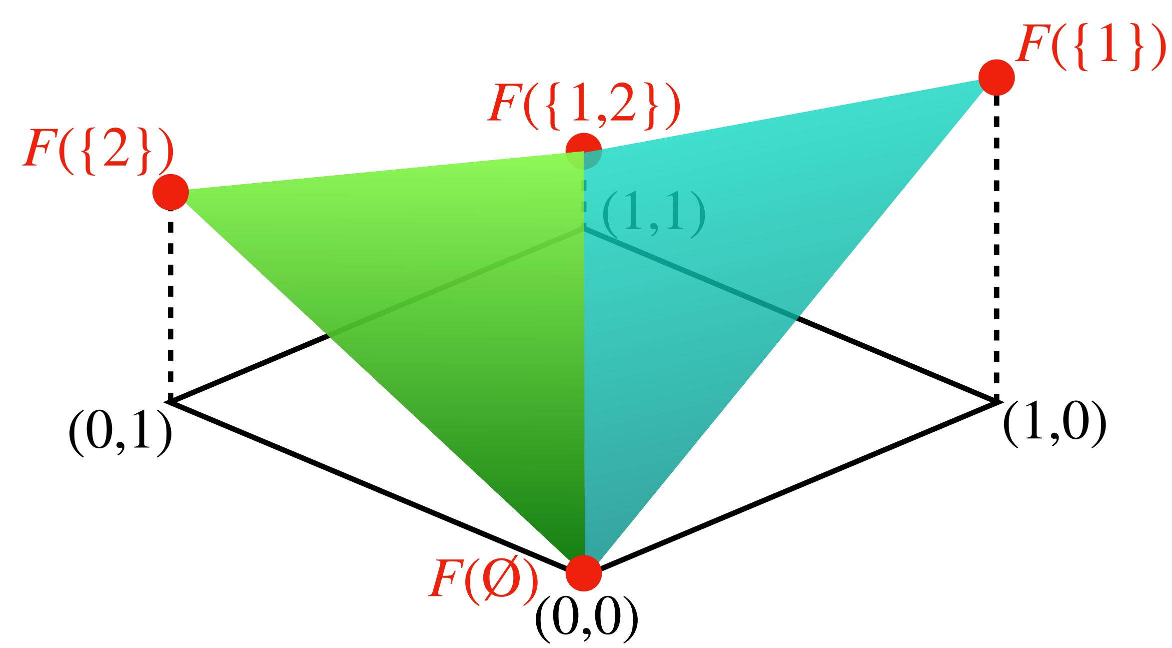

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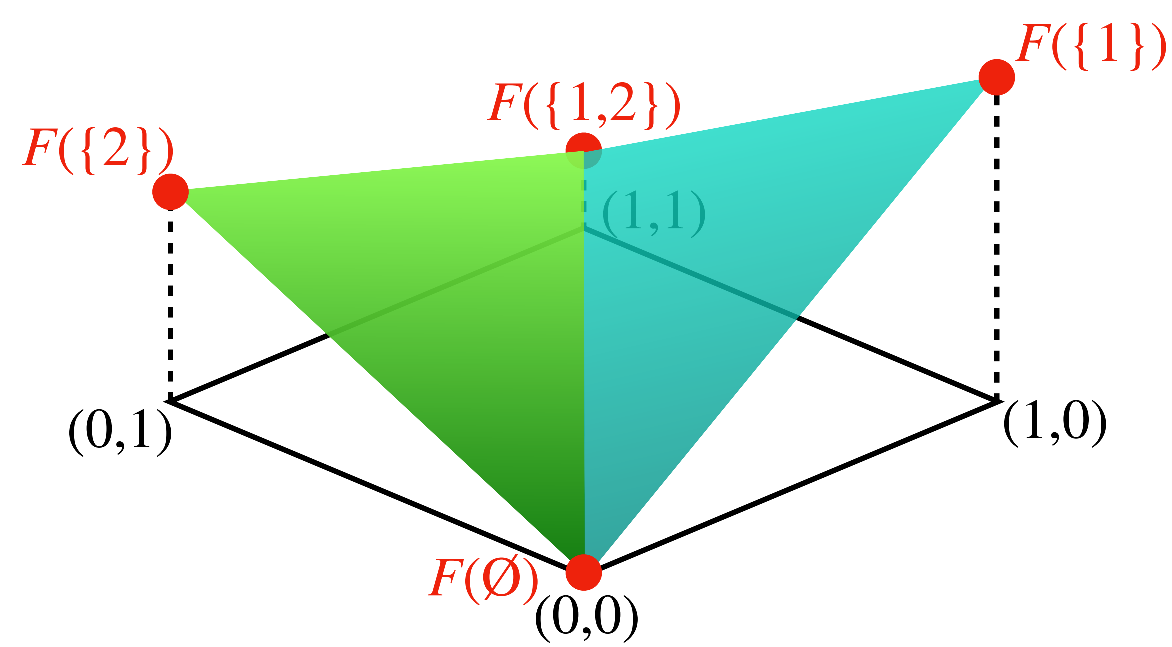

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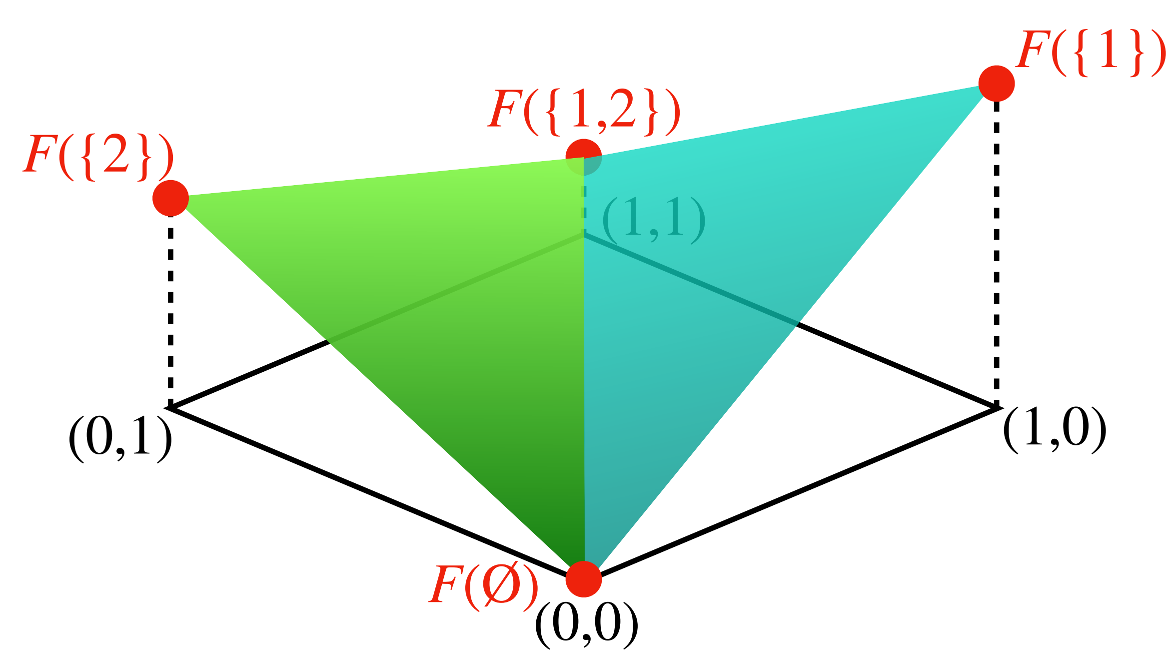

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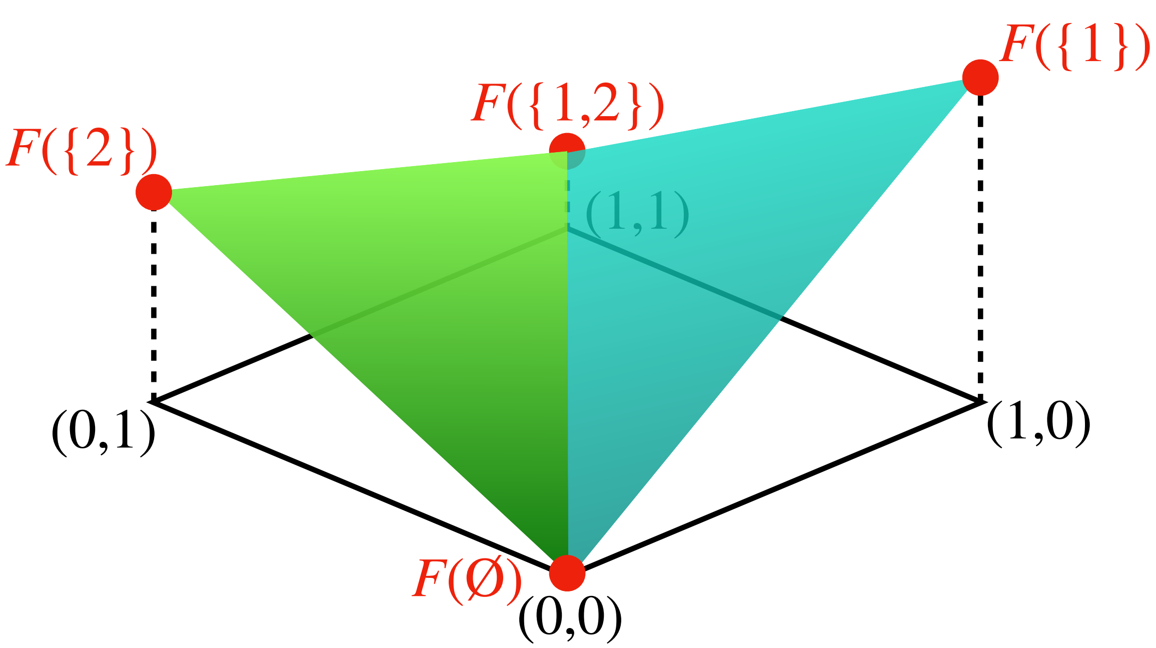
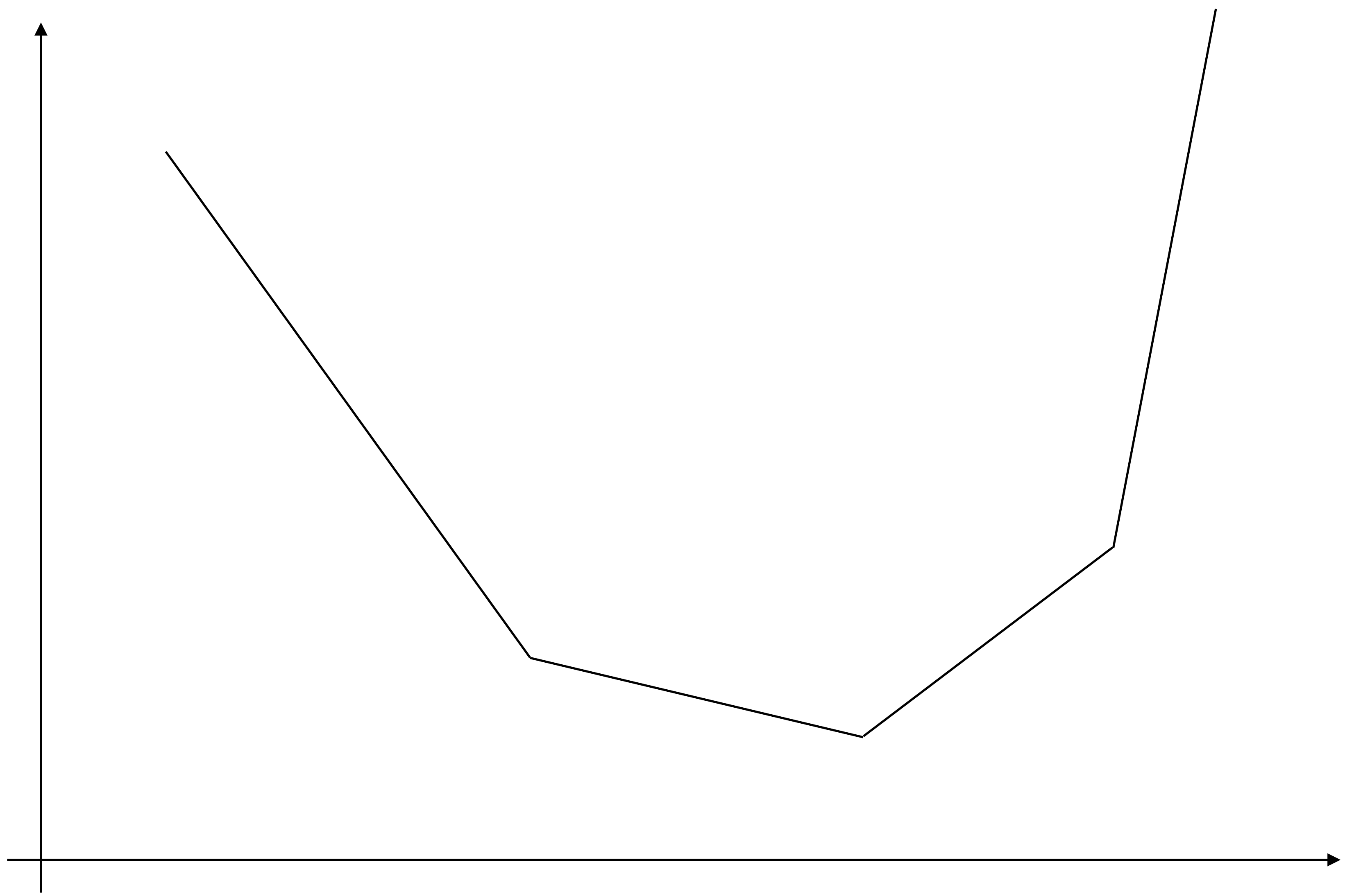


Stochastic Subgradient Descent

Convex function $f : C \to \mathbb{R}$ on a convex set C. *(not necessarily differentiable)*



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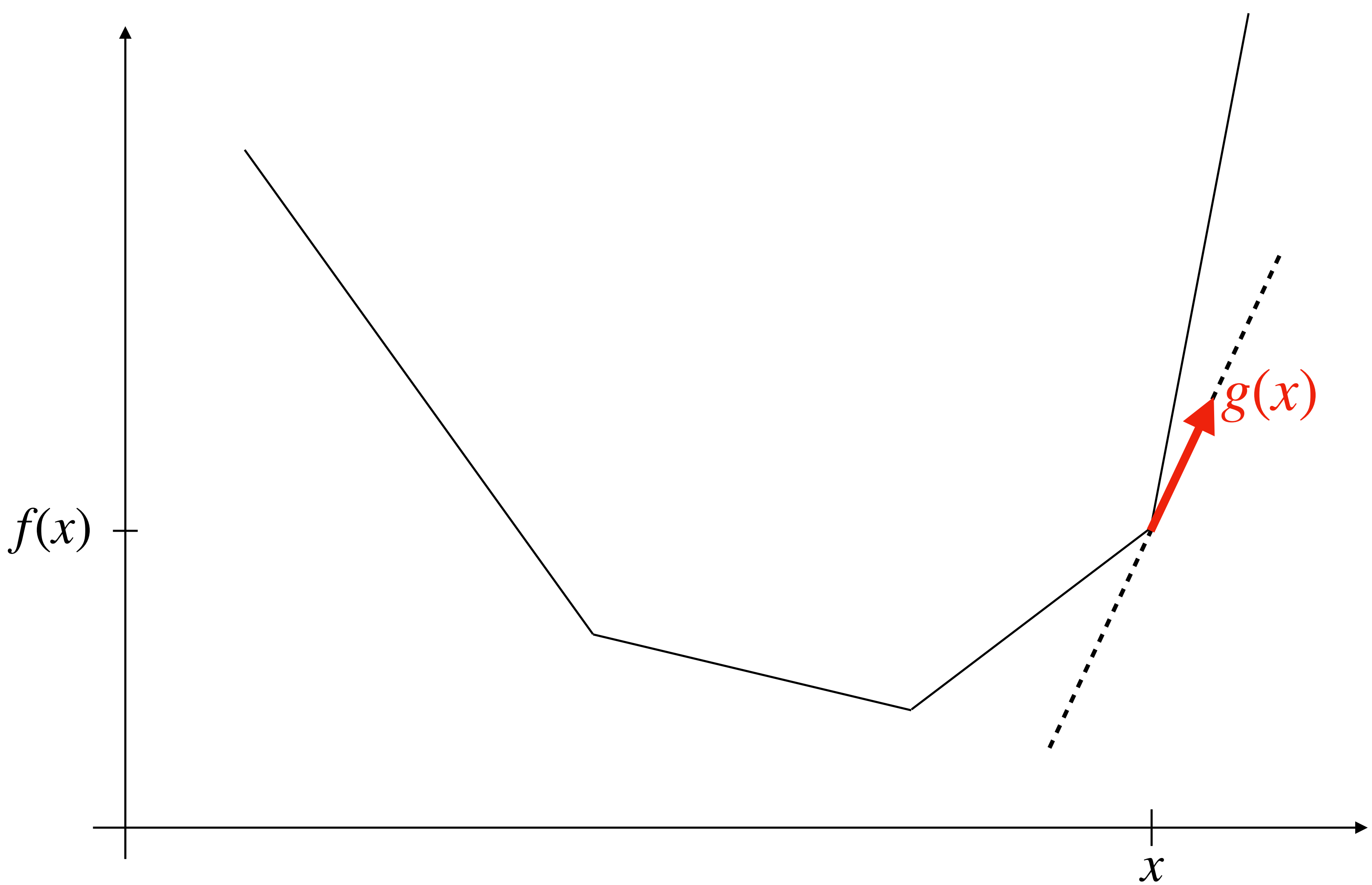

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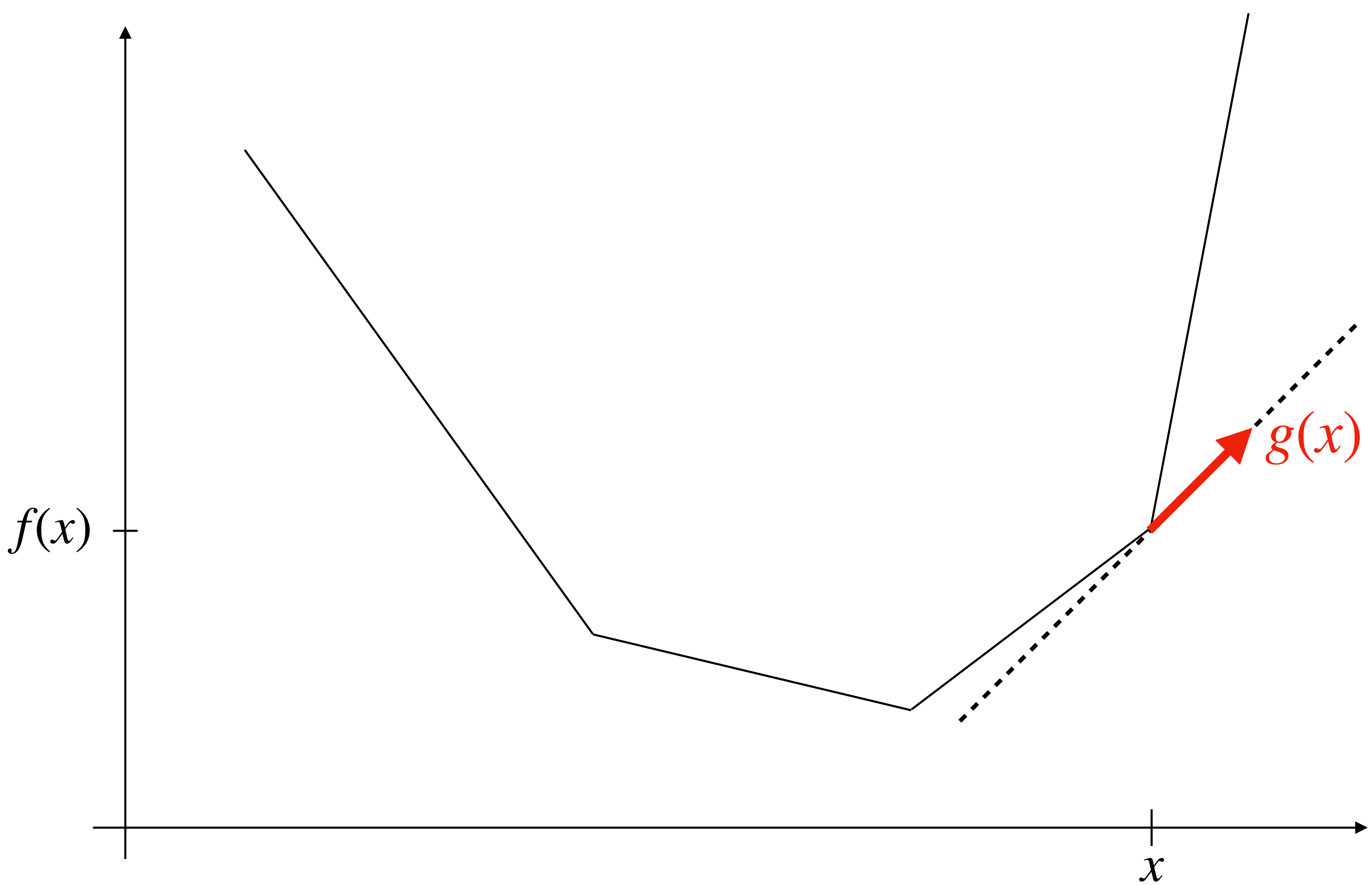

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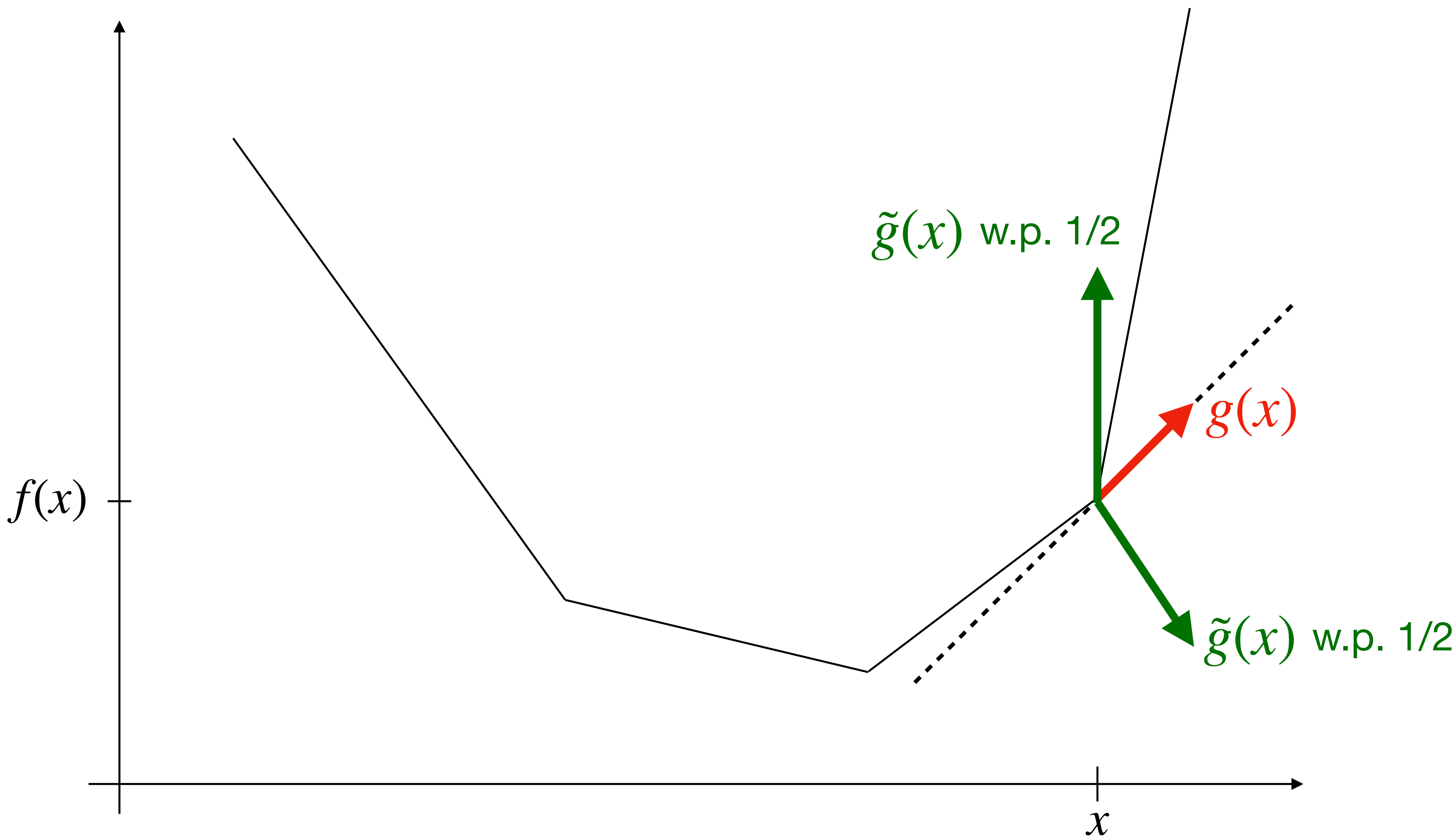

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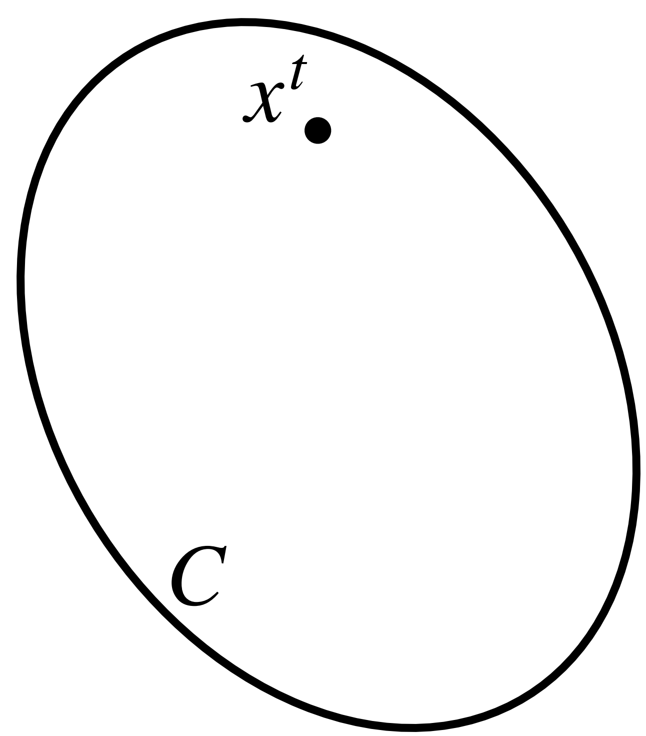

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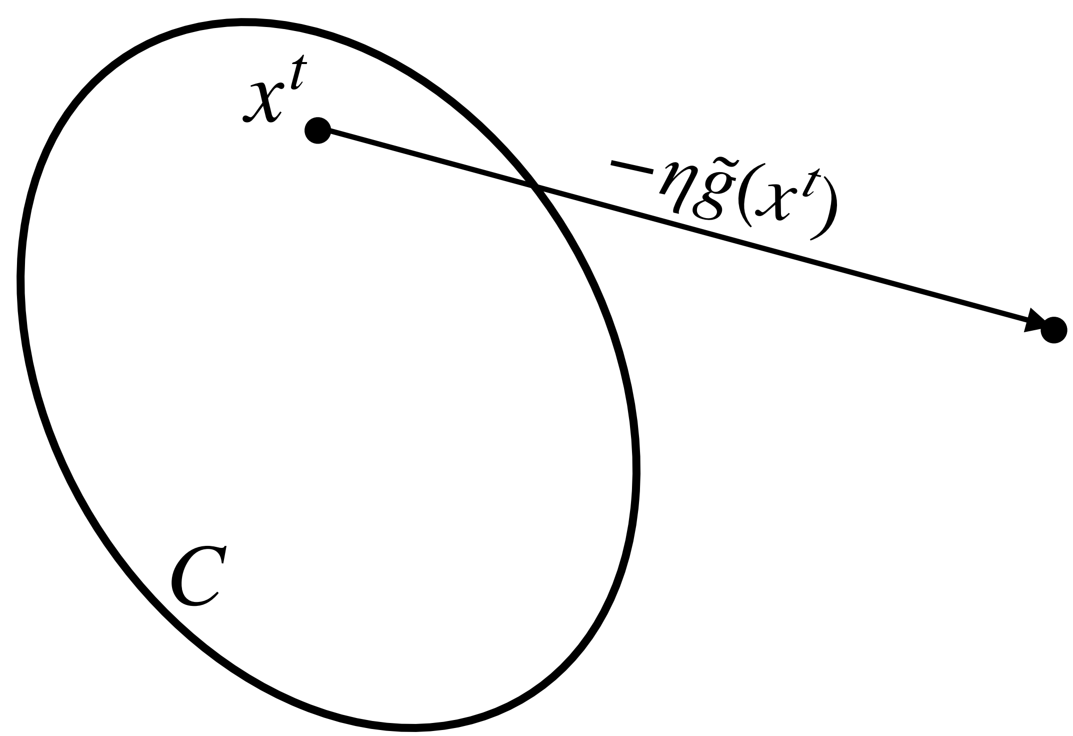

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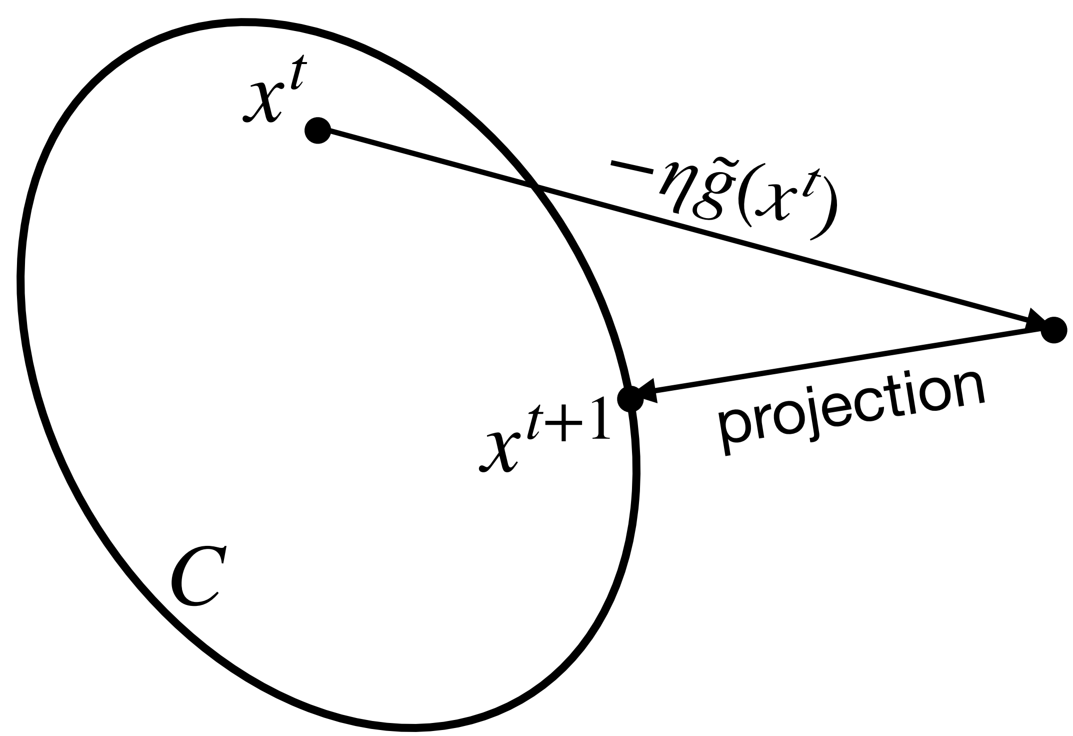

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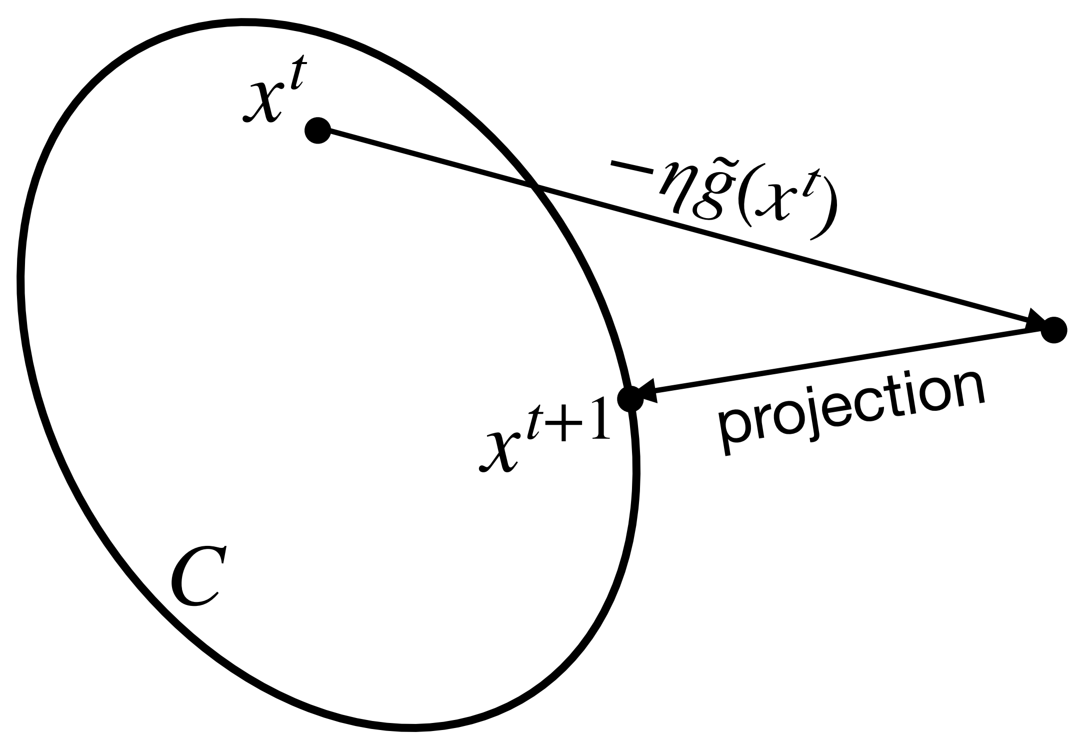

If $\tilde{g}(x)$ has **low variance** then the number of steps is the same as if we were using $g(x)$.

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This is where quantum computing comes in!

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Quantum speed-up for Importance Sampling

Problem

Input: discrete probability distribution $\mathbf{D = (p_1,\ldots,p_n)}$ on [n].

Output: **T** independent samples $i_1,\ldots,i_T \sim D$.

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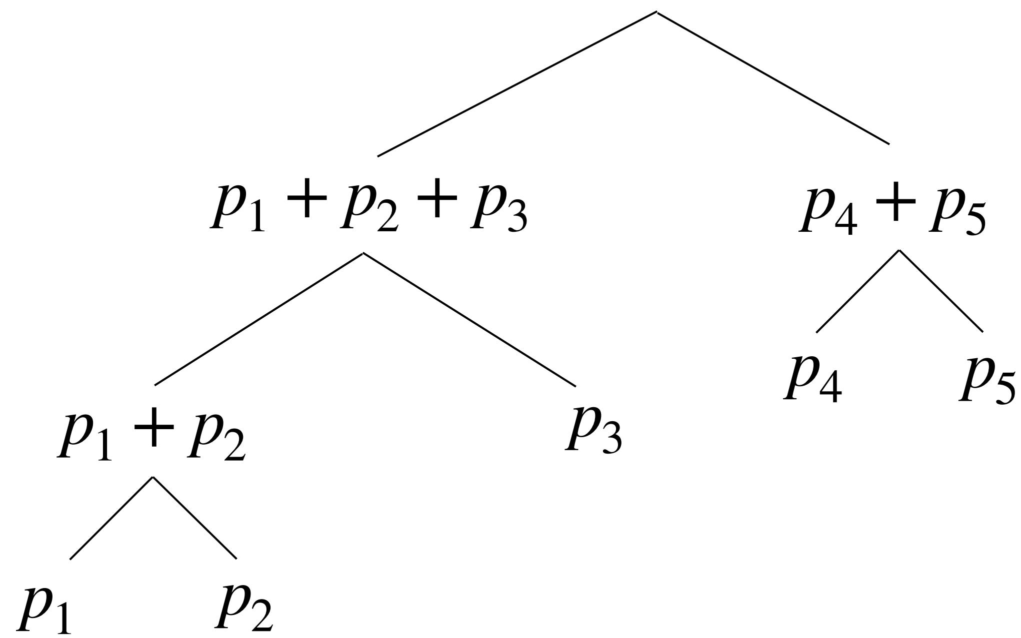
Can quantum computing help to sample faster?

Importance Sampling with a Binary Tree



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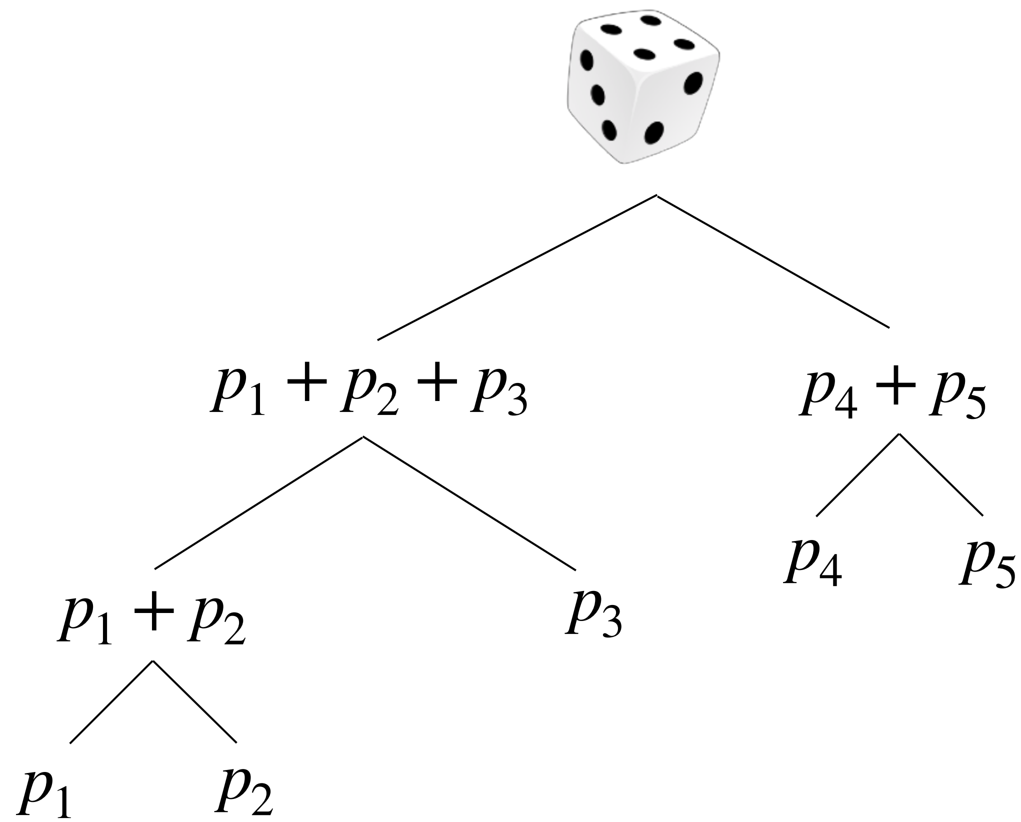

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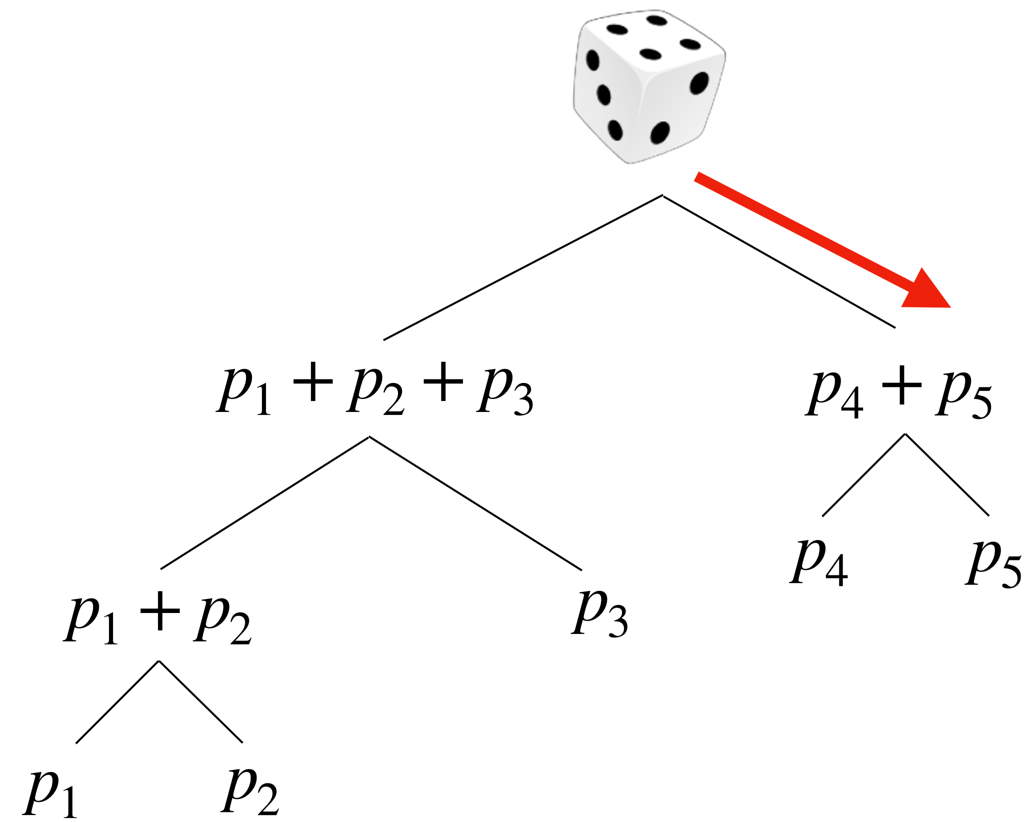

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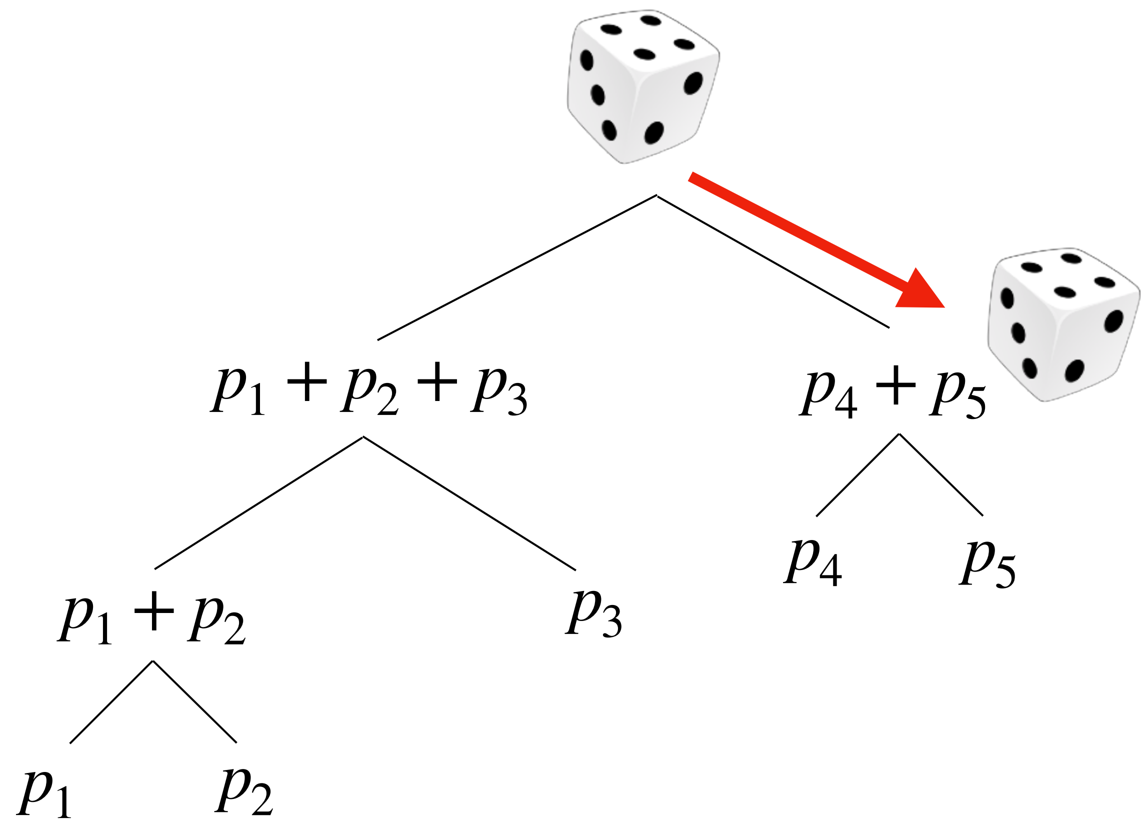

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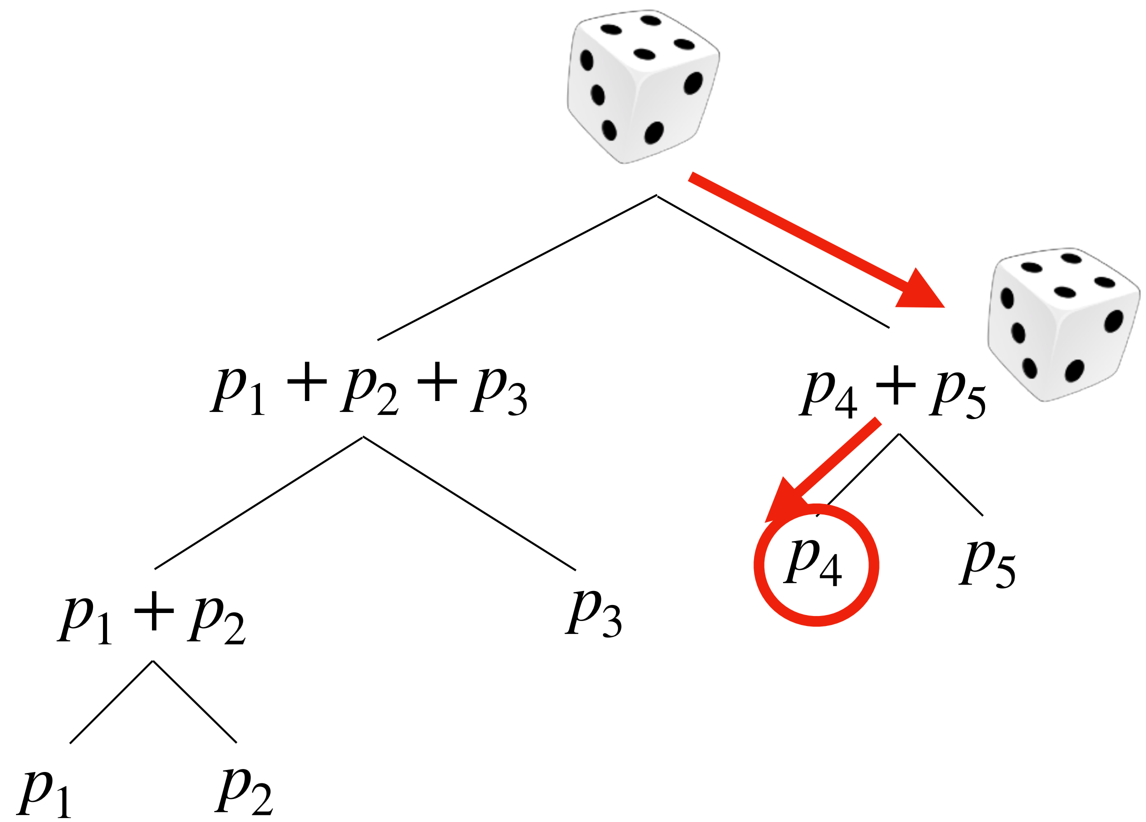

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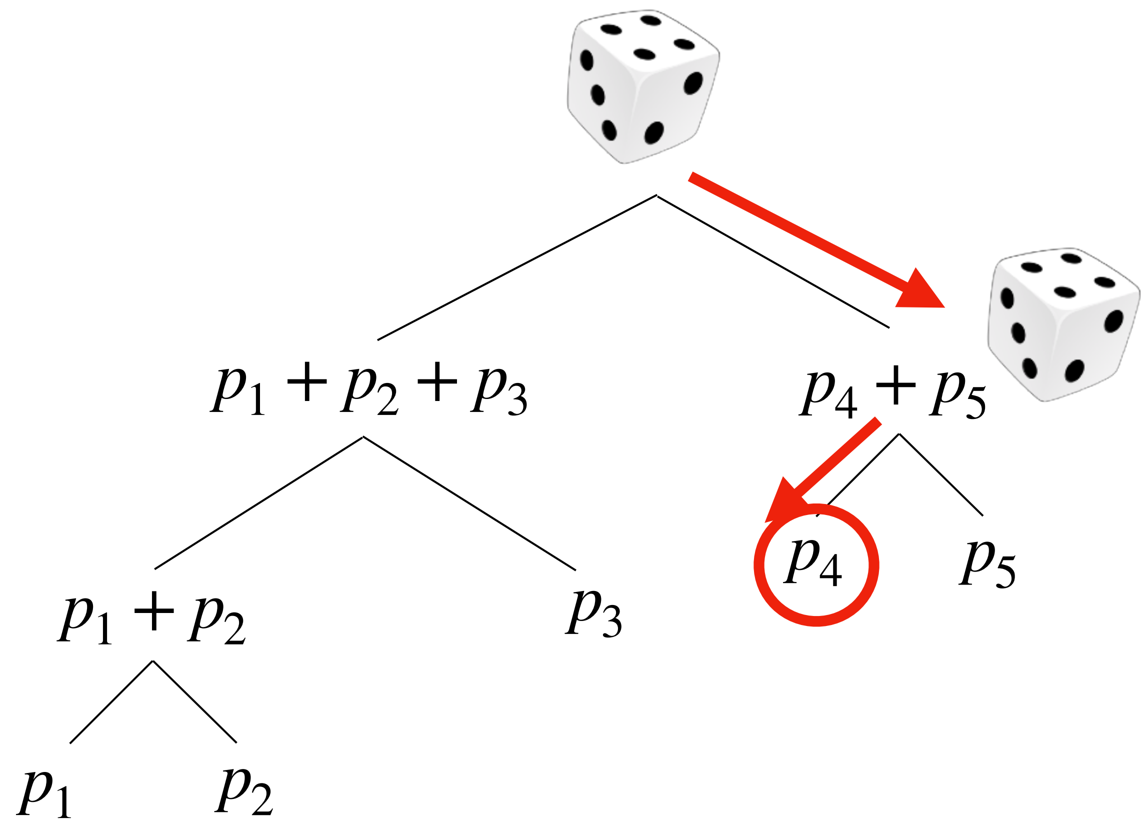

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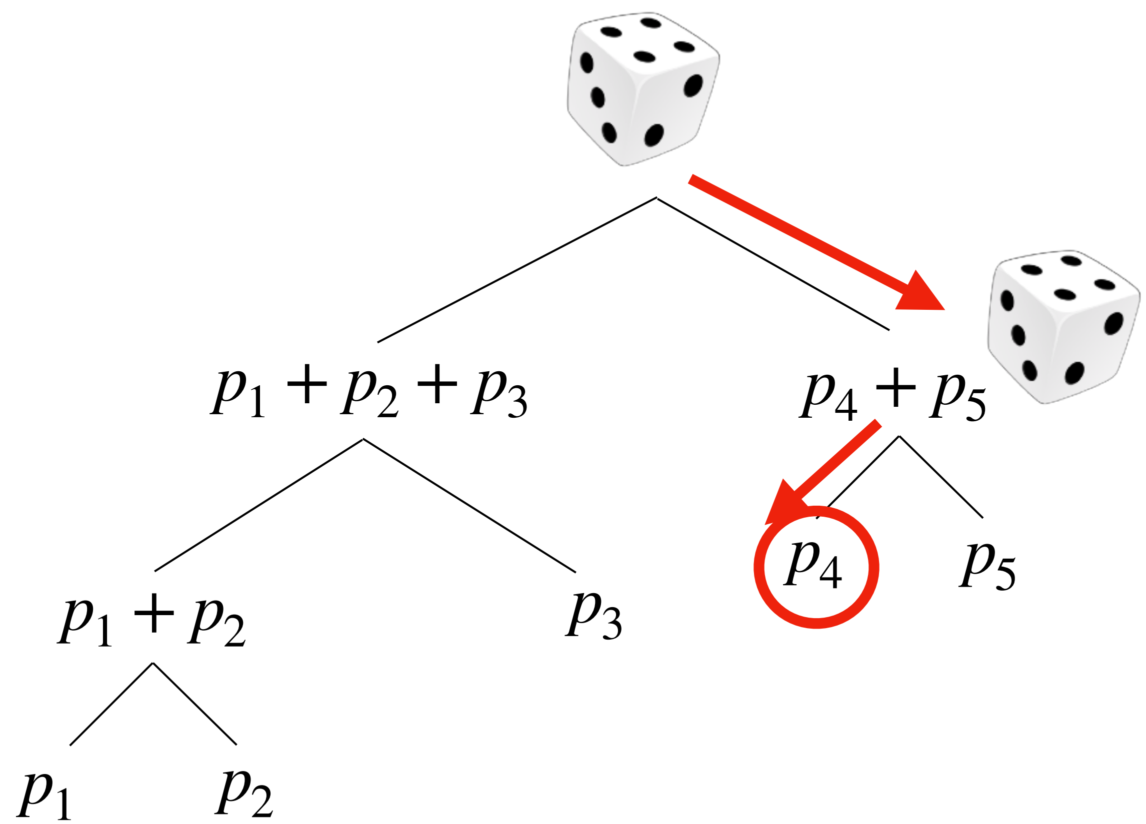

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Cost for T samples: $O(n + T \log n)$

Importance Sampling with Quantum State preparation

(Grover 2000)

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$$\longmapsto \frac{1}{\sqrt{n}} \sum_{i \in [n]} |i\rangle \left(\sqrt{\frac{p_i}{p_{\max}}}|0\rangle + \sqrt{1 - \frac{p_i}{p_{\max}}}|1\rangle \right)$$

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1. Compute $p_{\max} = \max\{p_1, \ldots, p_n\}$ with quantum **Maximum Finding**

2. Construct the unitary $V(|0\rangle|0\rangle) \longmapsto \frac{1}{\sqrt{n}} \sum_{i \in [n]} |i\rangle|0\rangle$

$$\longmapsto \frac{1}{\sqrt{n}} \sum_{i \in [n]} |i\rangle \left(\sqrt{\frac{p_i}{p_{\max}}} |0\rangle + \sqrt{1 - \frac{p_i}{p_{\max}}} |1\rangle \right)$$

$$= \frac{1}{\sqrt{n p_{\max}}} \left(\sum_i \sqrt{p_i} |i\rangle \right) |0\rangle + \ldots |1\rangle$$

Sampling **(repeat T times):**

Importance Sampling with Quantum State preparation

(Grover 2000)

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Sampling **(repeat T times):**

1. Prepare $\sum_i \sqrt{p_i} |i\rangle$ with **Amplitude Amplification** on V, and measure it.

Importance Sampling with Quantum State preparation

(Grover 2000)

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Cost per sample: $O(\sqrt{n p_{\max}})$

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Cost per sample: $O(\sqrt{n p_{\max}})$

Cost for T samples: $O(\sqrt{n} + T\sqrt{n p_{\max}})$

Importance Sampling with Quantum State preparation

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Sampling **(repeat T times)**:

1. Prepare $\sum_i \sqrt{p_i}|i\rangle$ with **Amplitude Amplification** on V, and measure it.

Preprocessing time: $O(\sqrt{n})$

Cost per sample: $O(\sqrt{n p_{\max}})$

Cost for T samples: $O(\sqrt{n} + T\sqrt{n p_{\max}}) = O(T\sqrt{n})$

Binary Tree

$$O(n + T \log n)$$

Quantum State Preparation

$$O(T\sqrt{n})$$

Importance Sampling

Binary Tree

$$O(n + T\log n)$$

Quantum State Preparation

$$O(T\sqrt{n})$$

For our submodular function minimization algorithm, we need **$T = \sqrt{n}$.**

Importance Sampling

Binary Tree

$O(n + T \log n)$

Quantum State Preparation

$O(T\sqrt{n})$

For our submodular function minimization algorithm, we need **T = √n.**

↓

New quantum multi-sampling algorithm in $O\left(\sqrt{Tn}\right)$

Importance Sampling with a Quantum Oracle

Our result: $O\left(\sqrt{Tn}\right)$ for obtaining T independent samples from D = $(p_1,\ldots,p_n)$.

Importance Sampling with a Quantum Oracle

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Element	1	2	3	4	5	6	7
Probability	p_1	p_2	p_3	p_4	p_5	p_6	p_7

Distribution D

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Distribution D

$p_i \geq 1/T$

$P_{\text{Heavy}} = \sum_{i \in \text{Heavy}} p_i$

Element	1	3	4
Probability	$\frac{p_1}{P_{\text{Heavy}}}$	$\frac{p_3}{P_{\text{Heavy}}}$	$\frac{p_4}{P_{\text{Heavy}}}$

Distribution D_{Heavy}

Importance Sampling with a Quantum Oracle

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Distribution D

$p_i \geq 1/T$

$P_{\text{Heavy}} = \sum_{i \in \text{Heavy}} p_i$

$p_i < 1/T$

$P_{\text{Light}} = \sum_{i \in \text{Light}} p_i$

Element	1	3	4
Probability	$\frac{p_1}{P_{\text{Heavy}}}$	$\frac{p_3}{P_{\text{Heavy}}}$	$\frac{p_4}{P_{\text{Heavy}}}$

Distribution D_{Heavy}

Element	2	5	6	7
Probability	$\frac{p_2}{P_{\text{Light}}}$	$\frac{p_5}{P_{\text{Light}}}$	$\frac{p_6}{P_{\text{Light}}}$	$\frac{p_7}{P_{\text{Light}}}$

Distribution D_{Light}

Importance Sampling with a Quantum Oracle

Our result: $O(\sqrt{Tn})$ for obtaining T independent samples from D = ($p_1,\ldots,p_n$).

Element	1	2	3	4	5	6	7
Probability	p_1	p_2	p_3	p_4	p_5	p_6	p_7

Distribution D

$p_i \geq 1/T$

$P_{\text{Heavy}} = \sum_{i \in \text{Heavy}} p_i$

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$P_{\text{Light}} = \sum_{i \in \text{Light}} p_i$

Element	1	3	4
Probability	$\frac{p_1}{P_{\text{Heavy}}}$	$\frac{p_3}{P_{\text{Heavy}}}$	$\frac{p_4}{P_{\text{Heavy}}}$

Distribution D_{Heavy}

Element	2	5	6	7
Probability	$\frac{p_2}{P_{\text{Light}}}$	$\frac{p_5}{P_{\text{Light}}}$	$\frac{p_6}{P_{\text{Light}}}$	$\frac{p_7}{P_{\text{Light}}}$

Distribution D_{Light}

Use a Binary Tree

Importance Sampling with a Quantum Oracle

Our result: $O(\sqrt{Tn})$ for obtaining T independent samples from D = (p_1,…,p_n).

Element	1	2	3	4	5	6	7
Probability	p_1	p_2	p_3	p_4	p_5	p_6	p_7

Distribution D

$p_i \geq 1/T$ $\quad P_{\text{Heavy}} = \sum_{i \in \text{Heavy}} p_i$

$p_i < 1/T$ $\quad P_{\text{Light}} = \sum_{i \in \text{Light}} p_i$

Element	1	3	4
Probability	$\frac{p_1}{P_{\text{Heavy}}}$	$\frac{p_3}{P_{\text{Heavy}}}$	$\frac{p_4}{P_{\text{Heavy}}}$

Distribution D_{Heavy}

Use a Binary Tree

Element	2	5	6	7
Probability	$\frac{p_2}{P_{\text{Light}}}$	$\frac{p_5}{P_{\text{Light}}}$	$\frac{p_6}{P_{\text{Light}}}$	$\frac{p_7}{P_{\text{Light}}}$

Distribution D_{Light}

Use Quantum State Preparation

Importance Sampling with a Quantum Oracle

Preprocessing:

Sampling **(repeat T times):**

Importance Sampling with a Quantum Oracle

Preprocessing:

1. Compute the set **Heavy** $\subset [n]$ of indices i such that $\mathbf{p_i \geq 1/T}$, using **Grover Search**.

Sampling (repeat T times):

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Preprocessing:

1. Compute the set **Heavy** ⊂ [n] of indices i such that **$p_i \geq 1/T$**, using **Grover Search**.

2. Compute **P_{Heavy}** $= \sum_{i \in \text{Heavy}} p_i$

Sampling **(repeat T times):**

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Preprocessing:

1. Compute the set **Heavy** ⊂ [n] of indices i such that $p_i \geq 1/T$, using **Grover Search**.

2. Compute $P_{\text{Heavy}} = \sum_{i \in \text{Heavy}} p_i$

3. Apply the preprocessing step of the **Binary Tree Method** on D_{Heavy}.

Sampling **(repeat T times):**

Importance Sampling with a Quantum Oracle

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1. Compute the set **Heavy** ⊂ [n] of indices i such that $p_i \geq 1/T$, using **Grover Search**.

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3. Apply the preprocessing step of the **Binary Tree Method** on D_{Heavy}.

4. Apply the preprocessing step of the **Quant. State Preparation** method on D_{Light}.

Sampling **(repeat T times):**

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Preprocessing:

1. Compute the set **Heavy** $\subset [n]$ of indices i such that $p_i \geq 1/T$, using **Grover Search**.

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Sampling (repeat T times)**:**

Flip a coin that is head with probability P_{Heavy} :

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Sampling (repeat T times):

Flip a coin that is head with probability P_{Heavy} :

- **Head**: sample $i \sim D_{Heavy}$ with the **Binary Tree Method.**

Importance Sampling with a Quantum Oracle

Preprocessing:

1. Compute the set **Heavy** ⊂ [n] of indices i such that $p_i \geq 1/T$, using **Grover Search**.
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Sampling (repeat T times):

Flip a coin that is head with probability P_{Heavy} :

- **Head**: sample $i \sim D_{Heavy}$ with the **Binary Tree Method.**
- **Tail**: sample $i \sim D_{Light}$ with **Quantum State Preparation.**

Importance Sampling with a Quantum Oracle

Preprocessing:

1. Compute the set **Heavy** $\subset [n]$ of indices i such that $\mathbf{p_i \geq 1/T}$, using **Grover Search**.

   Cost:

2. Compute $\mathbf{P_{Heavy}} = \sum_{i \in \mathrm{Heavy}} p_i$

   Cost:

3. Apply the preprocessing step of the **Binary Tree Method** on $\mathbf{D_{Heavy}}$.

   Cost:

4. Apply the preprocessing step of the **Quant. State Preparation** method on $\mathbf{D_{Light}}$.

   Cost:

Importance Sampling with a Quantum Oracle

Preprocessing:

1. Compute the set **Heavy** ⊂ [n] of indices i such that **$p_i \geq 1/T$**, using **Grover Search**.

   Cost: $O(\sqrt{nT})$ since **|Heavy| ≤ T**

2. Compute $\mathbf{P_{Heavy}} = \sum_{i \in \text{Heavy}} p_i$

   Cost:

3. Apply the preprocessing step of the **Binary Tree Method** on $\mathbf{D_{Heavy}}$.

   Cost:

4. Apply the preprocessing step of the **Quant. State Preparation** method on $\mathbf{D_{Light}}$.

   Cost:

Importance Sampling with a Quantum Oracle

Preprocessing:

1. Compute the set **Heavy** ⊂ [n] of indices i such that **$p_i \geq 1/T$**, using **Grover Search**.

   Cost: $O(\sqrt{nT})$ since **|Heavy| ≤ T**

2. Compute **P_{Heavy}** $= \sum_{i \in \text{Heavy}} p_i$

   Cost: $O(T)$

3. Apply the preprocessing step of the **Binary Tree Method** on **D_{Heavy}**.

   Cost:

4. Apply the preprocessing step of the **Quant. State Preparation** method on **D_{Light}**.

   Cost:

Preprocessing:

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   Cost: $O(T)$

4. Apply the preprocessing step of the **Quant. State Preparation** method on $\mathbf{D_{Light}}$.

   Cost: $O(\sqrt{n})$

Importance Sampling with a Quantum Oracle

Sampling **(repeat T times):**

Flip a coin that is head with probability P_{Heavy} :

- **Head**: sample $i \sim D_{Heavy}$ with the **Binary Tree Method.**

Cost per sample:

- **Tail**: sample $i \sim D_{Light}$ with **Quantum State Preparation.**

Importance Sampling with a Quantum Oracle

Sampling **(repeat T times):**

Flip a coin that is head with probability P_{Heavy} :

- **Head**: sample $i \sim D_{Heavy}$ with the **Binary Tree Method.**

Cost per sample: $O(\log n)$

- **Tail**: sample $i \sim D_{Light}$ with **Quantum State Preparation.**

Importance Sampling with a Quantum Oracle

Sampling **(repeat T times):**

Flip a coin that is head with probability **P_{Heavy}** :

- **Head**: sample **i ~ D_{Heavy}** with the **Binary Tree Method.**

Cost per sample: $O(\log n)$ **Total cost:** $O(T \log n)$

- **Tail**: sample **i ~ D_{Light}** with **Quantum State Preparation.**

Importance Sampling with a Quantum Oracle

Sampling **(repeat T times):**

Flip a coin that is head with probability **P_{Heavy}** :

- **Head**: sample **i ~ D_{Heavy}** with the **Binary Tree Method.**

Cost per sample: $O(\log n)$ **Total cost:** $O(T \log n)$

- **Tail**: sample **i ~ D_{Light}** with **Quantum State Preparation.**

Cost per sample:

Importance Sampling with a Quantum Oracle

Sampling **(repeat T times):**

Flip a coin that is head with probability **P_{Heavy}** :

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Cost per sample: $O(\log n)$ **Total cost:** $O(T \log n)$

- **Tail**: sample **$i \sim D_{Light}$** with **Quantum State Preparation.**

Cost per sample: $O\left(\sqrt{np_{\max}}\right)$ where $p_{\max} = \max\left\{\frac{p_i}{P_{\text{Light}}} : i \in \text{Light}\right\}$

Importance Sampling with a Quantum Oracle

Sampling **(repeat T times):**

Flip a coin that is head with probability **P_{Heavy}** :

- **Head**: sample **$i \sim D_{Heavy}$** with the **Binary Tree Method.**

Cost per sample: $O(\log n)$ **Total cost:** $O(T \log n)$

- **Tail**: sample **$i \sim D_{Light}$** with **Quantum State Preparation.**

Cost per sample: $O\left(\sqrt{np_{\max}}\right)$ where $p_{\max} = \max\left\{\frac{p_i}{P_{\text{Light}}} : i \in \text{Light}\right\} \leq \frac{1}{T \cdot P_{\text{Light}}}$

Importance Sampling with a Quantum Oracle

Sampling **(repeat T times):**

Flip a coin that is head with probability P_{Heavy} :

- **Head**: sample **i ~ D_{Heavy}** with the **Binary Tree Method.**

Cost per sample: $O(\log n)$ **Total cost:** $O(T \log n)$

- **Tail**: sample **i ~ D_{Light}** with **Quantum State Preparation.**

Cost per sample: $O\left(\sqrt{np_{\max}}\right)$ where $p_{\max} = \max\left\{\frac{p_i}{P_{\text{Light}}} : i \in \text{Light}\right\} \leq \frac{1}{T \cdot P_{\text{Light}}}$

Total expected cost:

Sampling **(repeat T times):**

Flip a coin that is head with probability **P_{Heavy}** :

- **Head**: sample **i ~ D_{Heavy}** with the **Binary Tree Method.**

Cost per sample: $O(\log n)$ **Total cost:** $O(T \log n)$

- **Tail**: sample **i ~ D_{Light}** with **Quantum State Preparation.**

Cost per sample: $O\left(\sqrt{np_{\max}}\right)$ where $p_{\max} = \max\left\{\frac{p_i}{P_{\text{Light}}} : i \in \text{Light}\right\} \leq \frac{1}{T \cdot P_{\text{Light}}}$

Total expected cost: $O\left(T \cdot P_{\text{Light}} \cdot \sqrt{np_{\max}}\right)$

Importance Sampling with a Quantum Oracle

Sampling **(repeat T times):**

Flip a coin that is head with probability P_{Heavy} :

- **Head**: sample $i \sim D_{Heavy}$ with the **Binary Tree Method.**

Cost per sample: $O(\log n)$ **Total cost:** $O(T \log n)$

- **Tail**: sample $i \sim D_{Light}$ with **Quantum State Preparation.**

Cost per sample: $O\left(\sqrt{np_{\max}}\right)$ where $p_{\max} = \max\left\{\frac{p_i}{P_{\text{Light}}} : i \in \text{Light}\right\} \leq \frac{1}{T \cdot P_{\text{Light}}}$

Total expected cost: $O\left(T \cdot P_{\text{Light}} \cdot \sqrt{np_{\max}}\right) = O\left(\sqrt{n \cdot T \cdot P_{\text{Light}}}\right)$

Importance Sampling with a Quantum Oracle

Sampling (repeat T times):

Flip a coin that is head with probability **P_{Heavy}** :

- **Head**: sample **$i \sim D_{Heavy}$** with the **Binary Tree Method.**

Cost per sample: $O(\log n)$ **Total cost:** $O(T \log n)$

- **Tail**: sample **$i \sim D_{Light}$** with **Quantum State Preparation.**

Cost per sample: $O\left(\sqrt{np_{\max}}\right)$ where $p_{\max} = \max\left\{\frac{p_i}{P_{\text{Light}}} : i \in \text{Light}\right\} \leq \frac{1}{T \cdot P_{\text{Light}}}$

Total expected cost: $O\left(T \cdot P_{\text{Light}} \cdot \sqrt{np_{\max}}\right) = O\left(\sqrt{n \cdot T \cdot P_{\text{Light}}}\right) = O\left(\sqrt{nT}\right)$

Conclusion

Recent improvement:

- Axelrod, Liu, Sidford 2019: classical $\tilde{O}(n/\epsilon^2)$ algorithm for approximate submodular function minimization

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- Axelrod, Liu, Sidford 2019: classical $\tilde{O}(n/\epsilon^2)$ algorithm for approximate submodular function minimization

Open questions:

- Can we improve the upper/lower bounds for exact/approximate submodular function minimization?

- What are other applications of our quantum multi-sampling algorithm? (*ongoing work: solving linear systems)*

- Can we prepare T copies of the state $\sum_{i \in [n]} \sqrt{p_i} \, | \, i \rangle$ in time $O\big(\sqrt{nT}\big)$.

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