A Sublinear-Time Quantum Algorithm for Approximating Partition Functions

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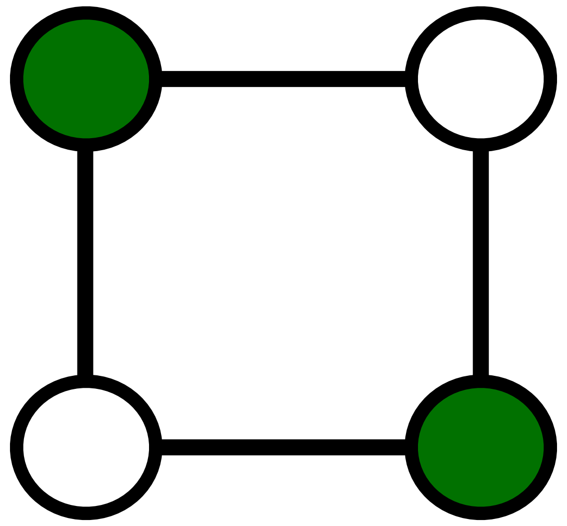
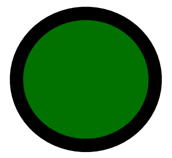
Independent set

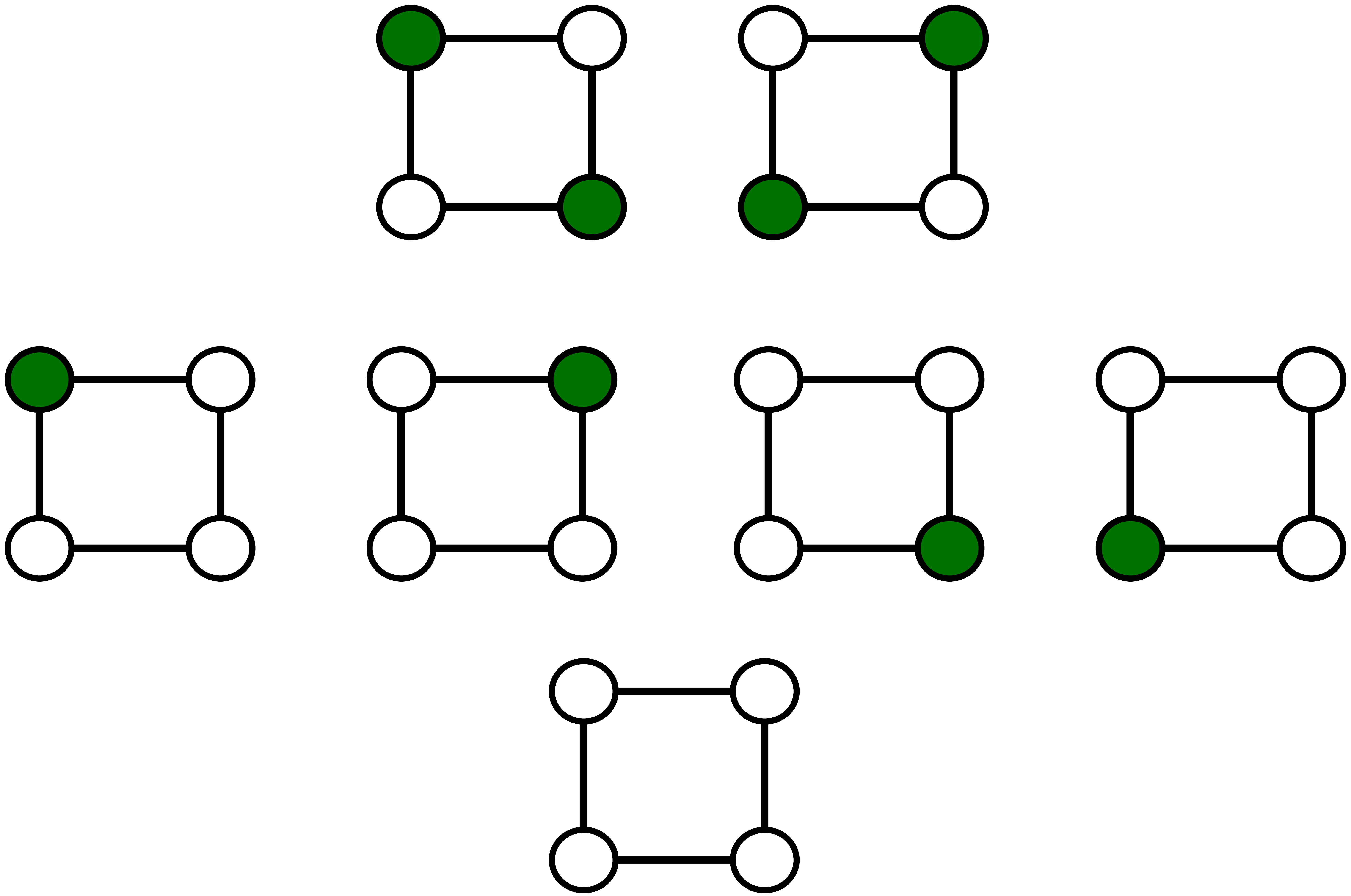
= subset of non-adjacent vertices





Hard-core gas model in statistical physics

independent sets = 7



Count # independent sets

#P-hard in many regimes

Bipartite graphs
[Provan,Ball’83]

3-regular graphs
[Dyer,Greenhill'00]

...

Exact counting → Approximate counting?

Approximate # independent sets

$$(1-\epsilon)\ \#\text{ind} \leq S \leq (1+\epsilon)\ \#\text{ind}$$

Classical algorithms

- $\tilde{O}(n^2/\epsilon^2)$ [Štefankovič,Vempala,Vigoda'09] [Chen,Liu,Vigoda'21]
- 5
- No FPRAS unless NP = RP [Sly'10]

Maximum vertex degree

Quantum algorithms

- $\tilde{O}(n^2 + n^{3/2}/\epsilon)$ [Montanaro'15]
- $\tilde{O}(n^{3/2}/\epsilon)$ [Harrow,Wei'20]
- $\tilde{O}(n^{5/4}/\epsilon)$ **Our work**

$n = \#\text{vertices}$

Weighted independent sets

$Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right)$

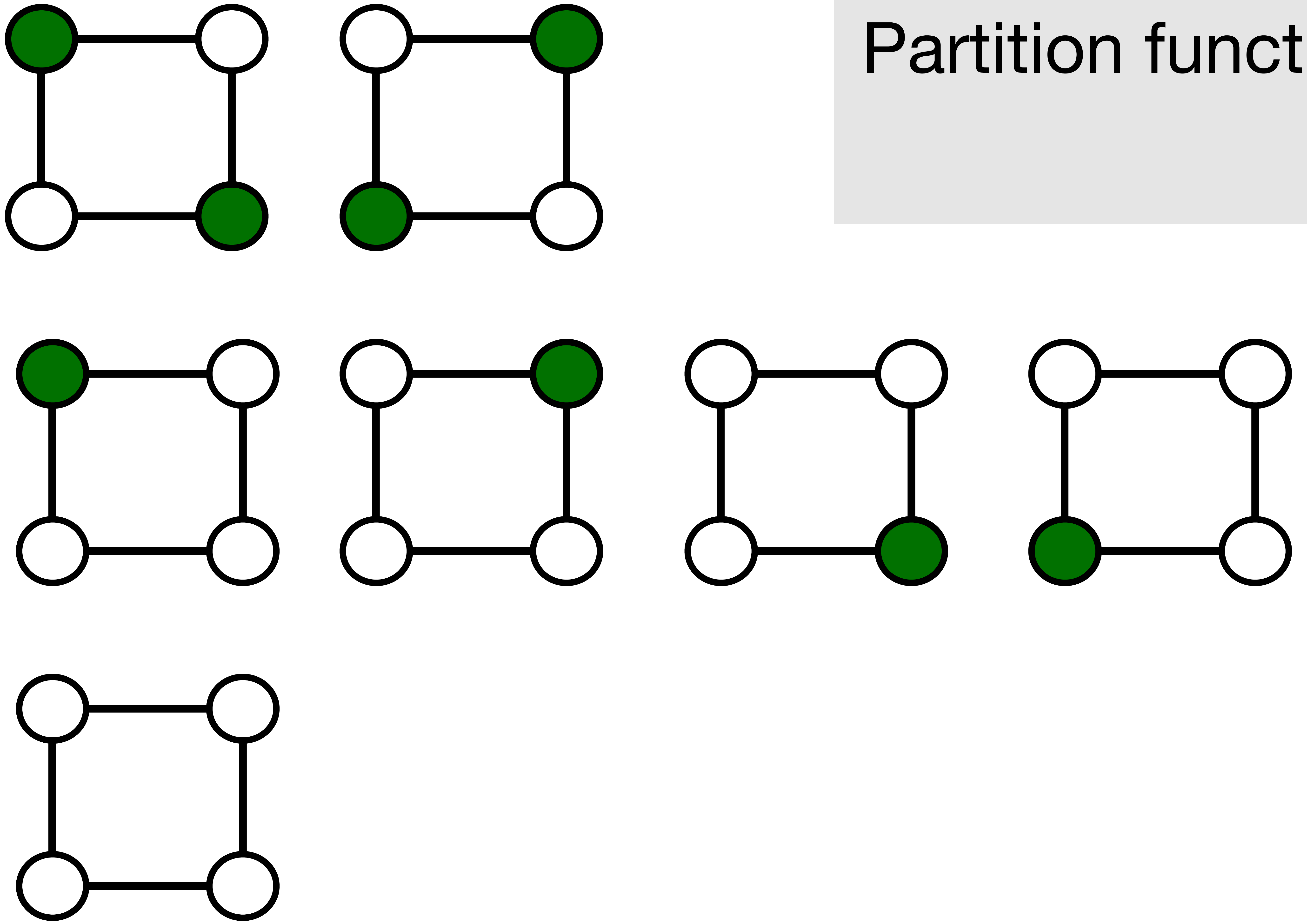
λ = fugacity

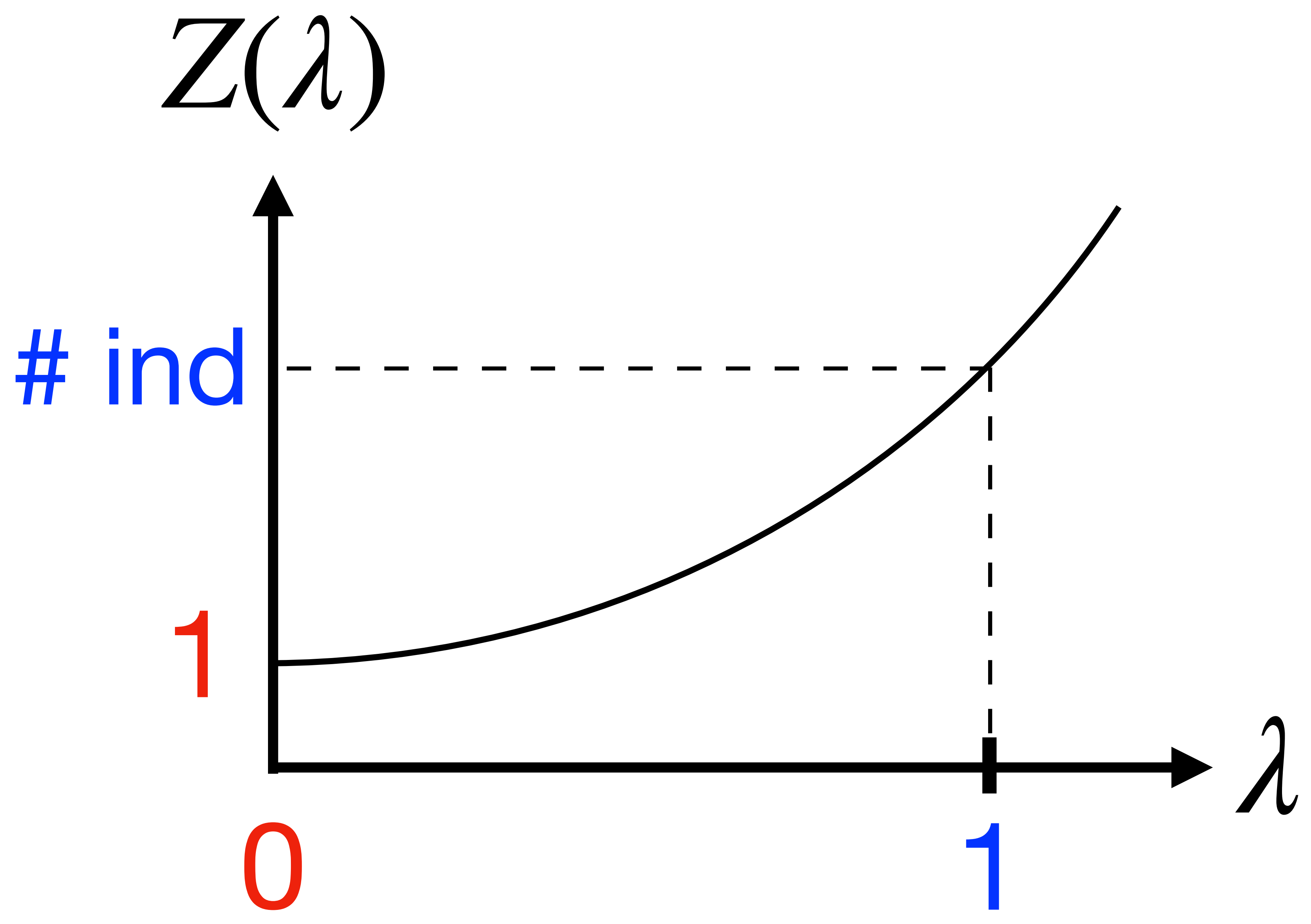
Partition function: $Z(\lambda) = \sum_{I \text{ ind. set}} \lambda^{|I|}$

λ^2

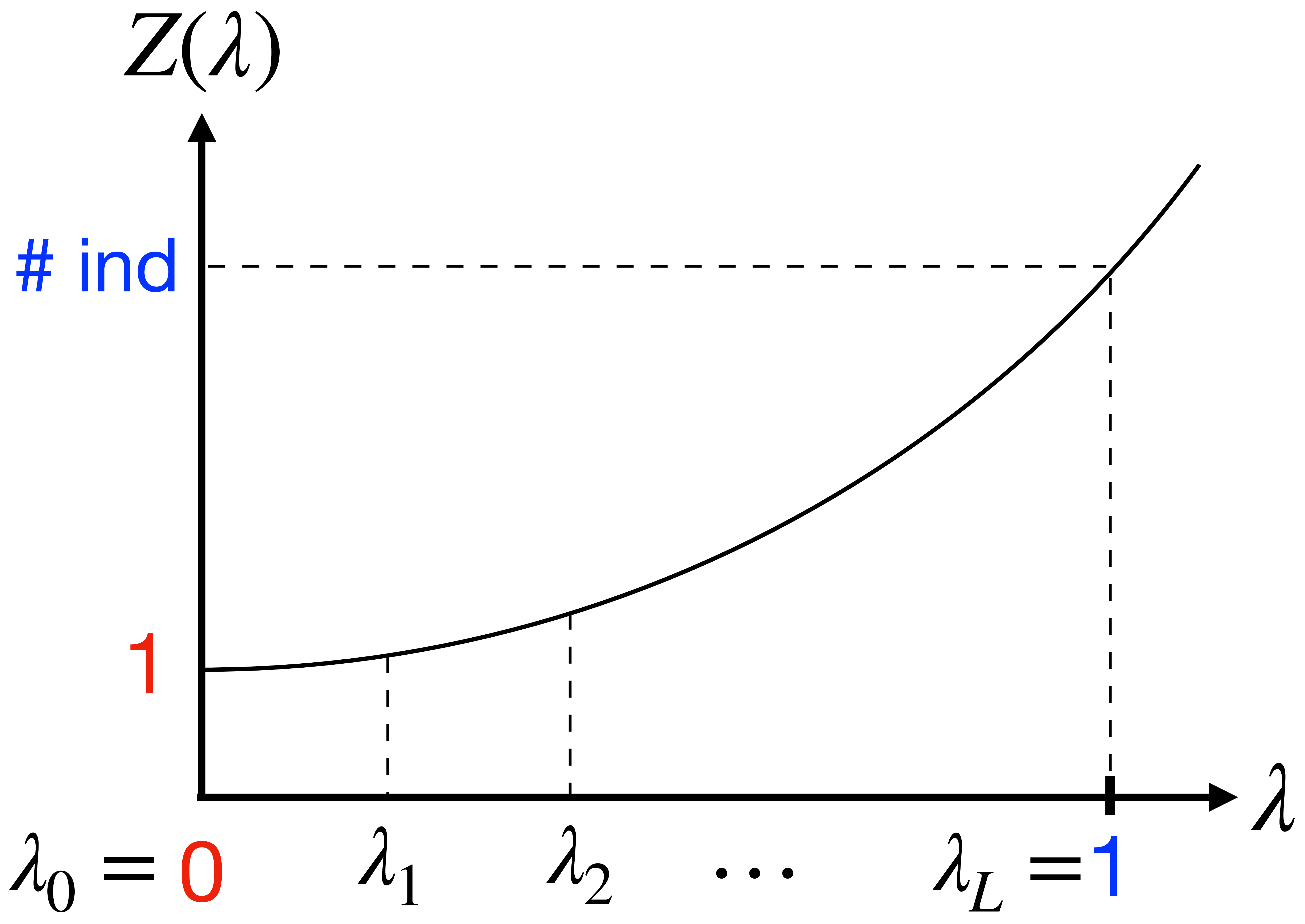
λ

1





Markov Chain Monte Carlo



Partition function: $Z(\lambda) = \sum_{I \text{ ind. set}} \lambda^{|I|}$

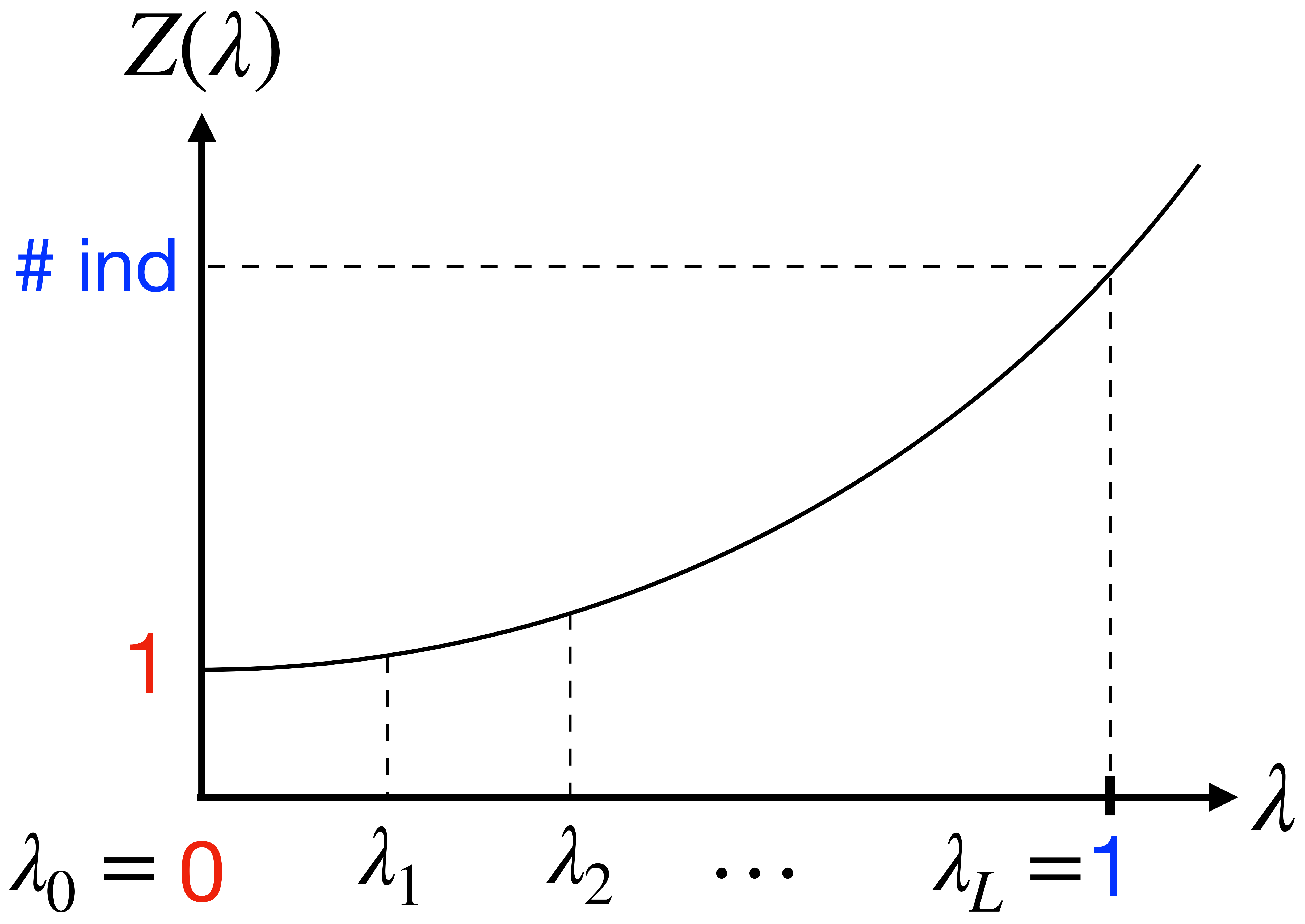
Gibbs distribution: $\pi_\lambda(I) = \dfrac{\lambda^{|I|}}{Z(\lambda)}$

$$Z(\lambda_{k+1}) = Z(\lambda_k) \cdot \mathrm{E}_{I \sim \pi_{\lambda_k}} \left(\frac{\lambda_{k+1}}{\lambda_k} \right)^{|I|}$$

→

$$Z(\lambda_{k+1}) = Z(\lambda_k) \cdot \mathrm{E}_{\pi_k}(X_k)$$

Markov Chain Monte Carlo



$$Z(1) = \mathrm{E}_{\pi_0}(X_0) \cdot \mathrm{E}_{\pi_1}(X_1) \cdots \mathrm{E}_{\pi_{L-1}}(X_{L-1})$$

Fast mixing Markov chain for π_k

Classical: Glauber dynamics [Chen,Liu,Vigoda'21]

Quant. speedup: Szegedy quantum walk + quant. simulated annealing [Wocjan,Abeyesinghe'08]

Quantum sample: $|\pi_k\rangle = \sum_I \sqrt{\pi_k(I)}\,|I\rangle$

Sample efficient estimator for $\mathrm{E}_{\pi_k}(X_k)$

Classical: Empirical mean

Quant. speedup: Based on Phase Estimation [Montanaro'15], [Harrow,Wei'20]

Our improvement: Unbiased estimator

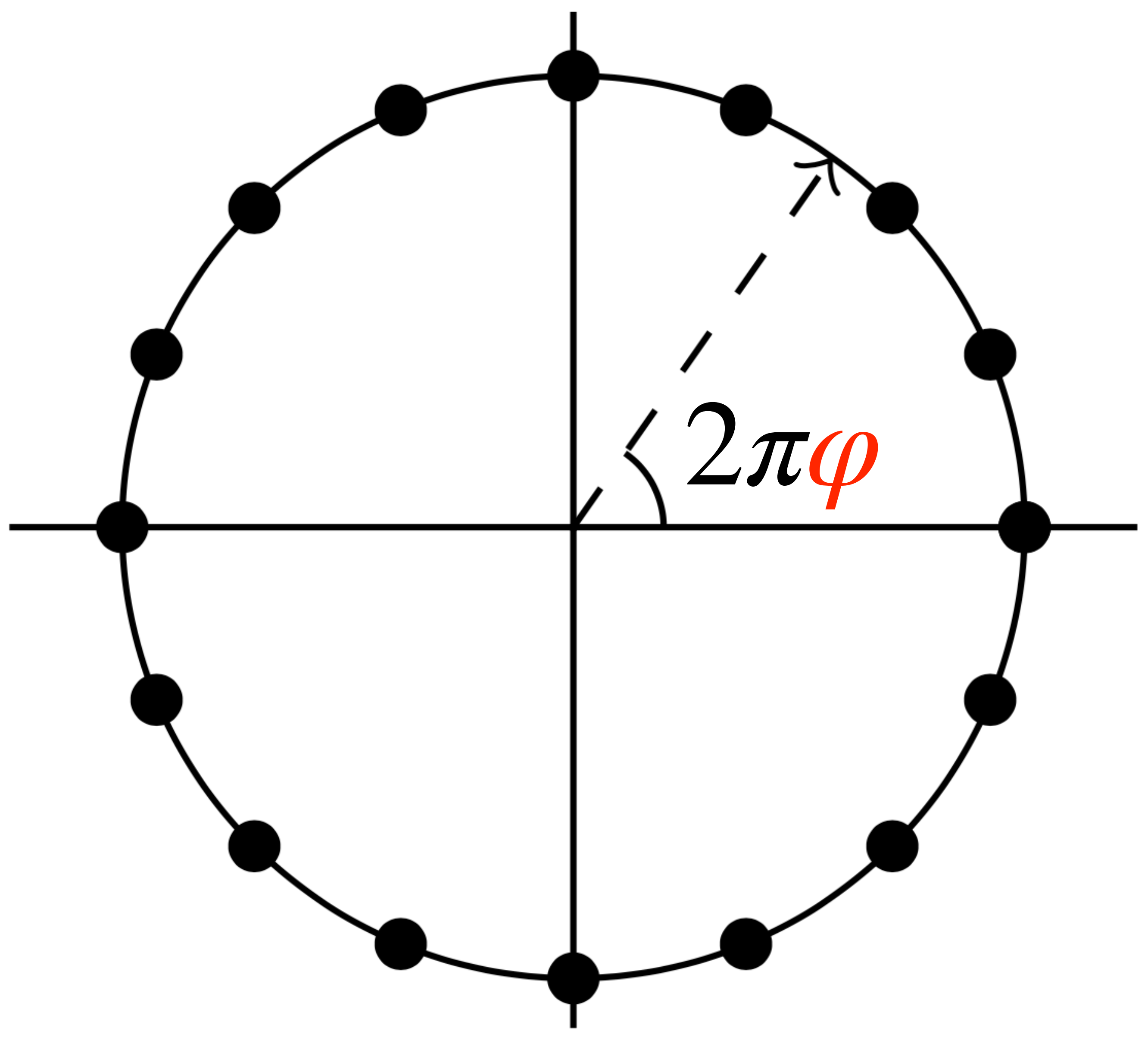
$\Rightarrow$ Better product estimate $S_0 \cdots S_{L-1} \approx Z(1)$

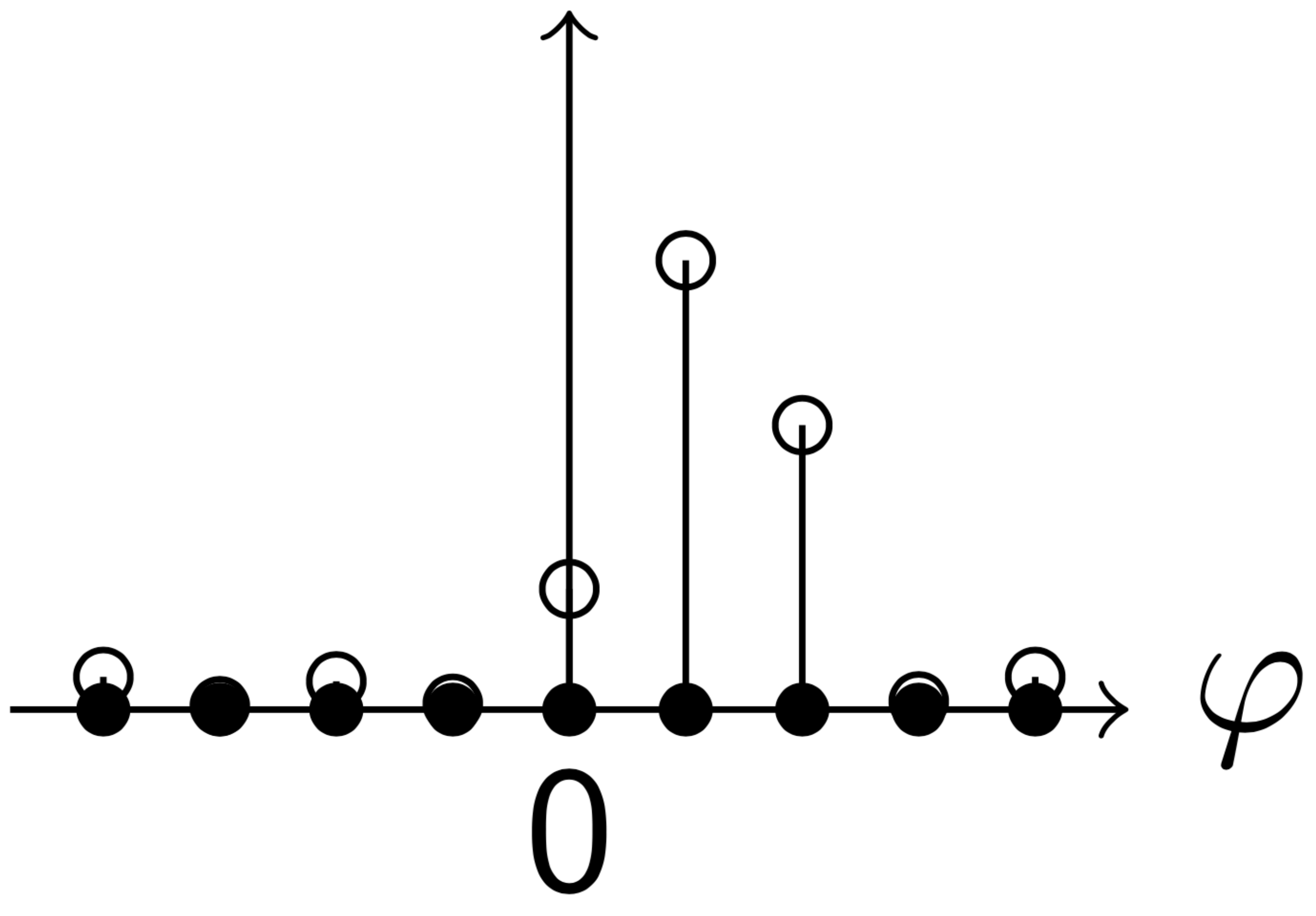
Unbiased Phase Estimation

Estimate φ where $U|\pi\rangle = e^{2\pi i\varphi}|\pi\rangle$

Standard approach

⇒ Biased finite outcome set





Unbiased Phase Estimation

Estimate φ where $U|\pi\rangle = e^{2\pi i\varphi}|\pi\rangle$

Standard approach

$\Rightarrow$ Biased finite outcome set

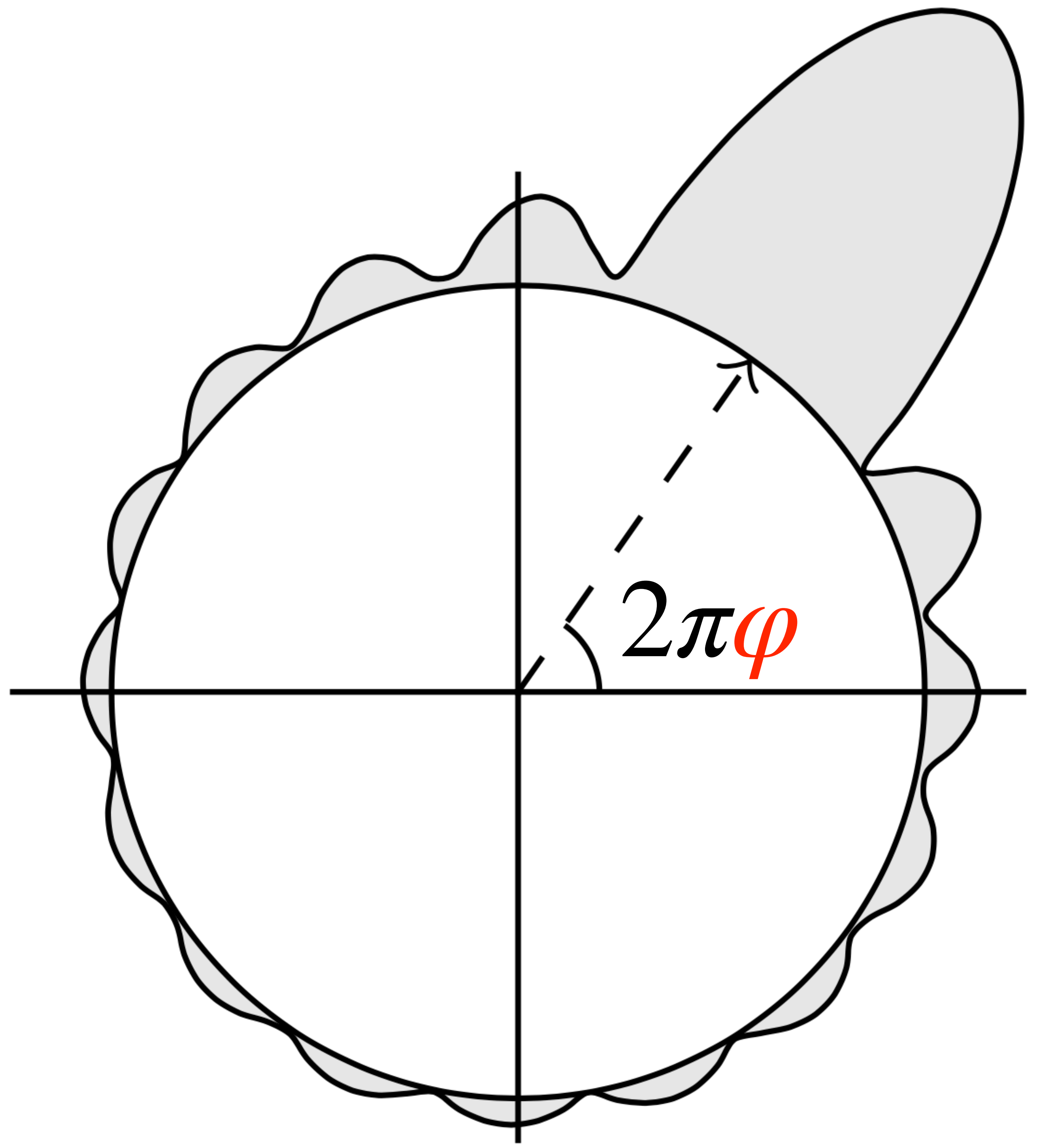
Symmetrization

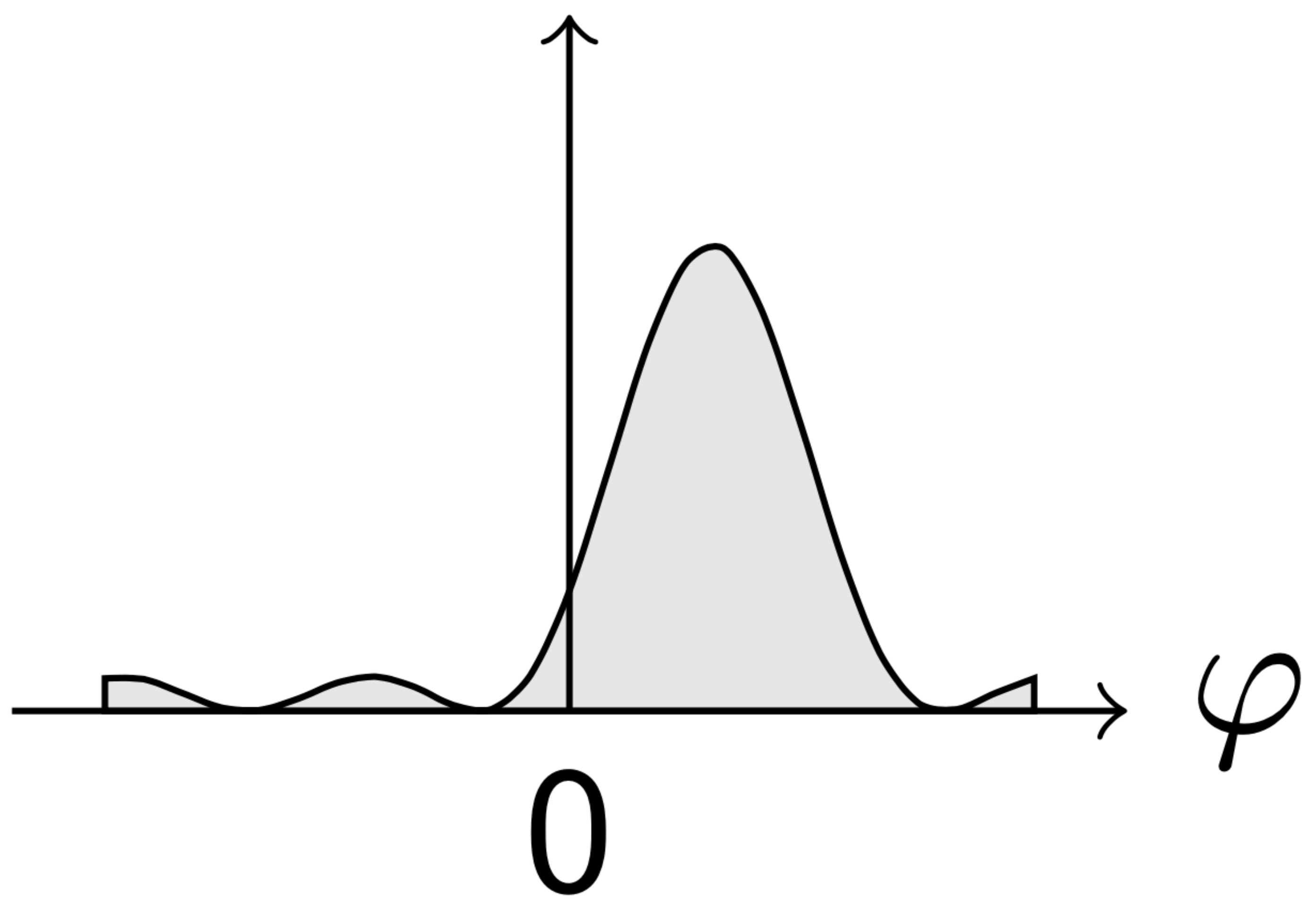
1) Sample a random phase θ

2) Run Phase Est. on $e^{2\pi i\theta}U$

3) Correct for choice of θ

$\Rightarrow$ Unbiased estimate of $e^{2\pi i\varphi}$





Unbiased Probability Estimation

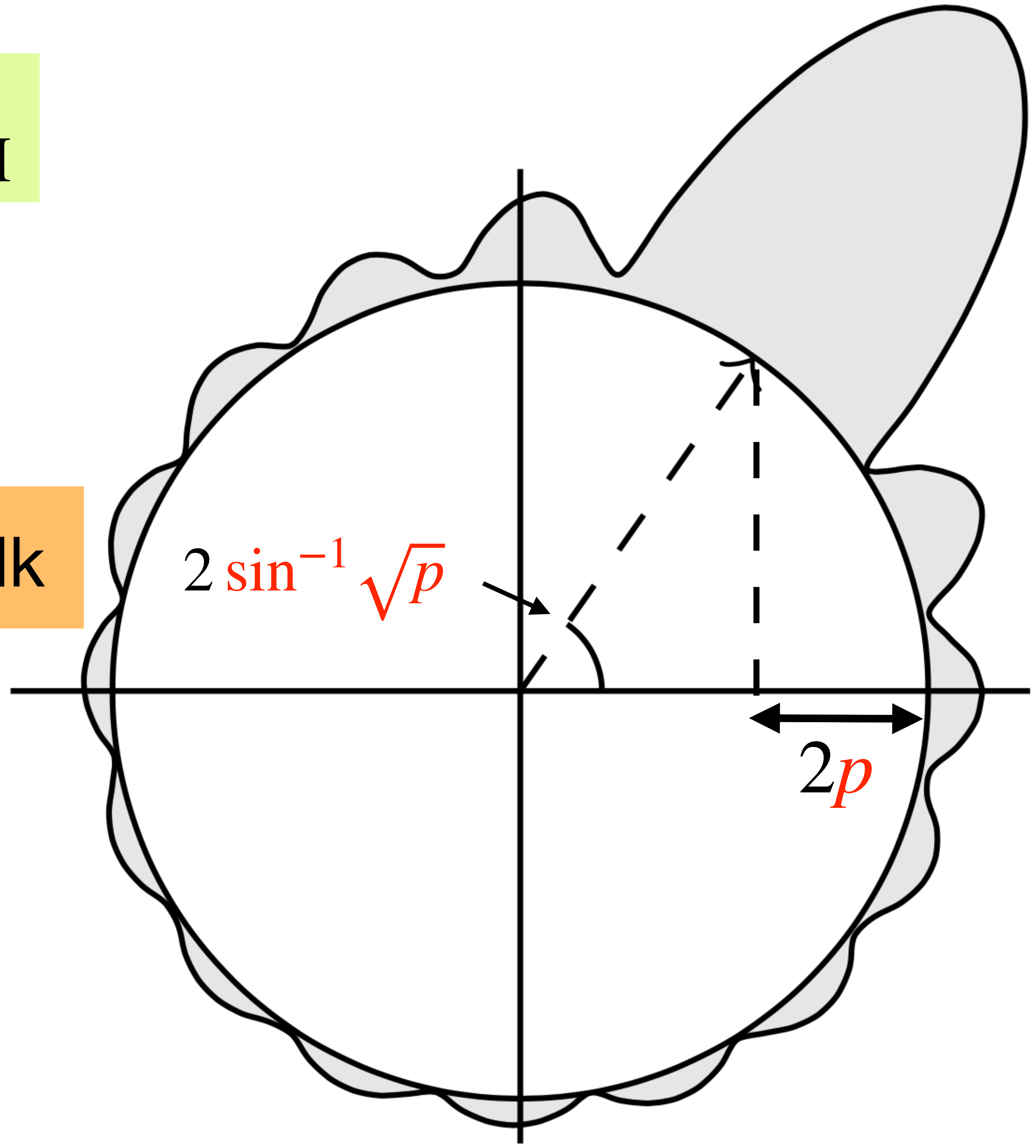
Estimate $p = \|\Pi\,|\pi\rangle\|^2$ for a projector Π

Unbiased Phase Estimation on $U = \mathrm{Ref}_\pi \mathrm{Ref}_\Pi$

Unbiased $p = \frac{1}{2}\left(1 - \mathrm{Re}\left[e^{2i \sin^{-1}\sqrt{p}}\right]\right)$

Restoring $|\pi\rangle$?

Needed to warm-start next walk



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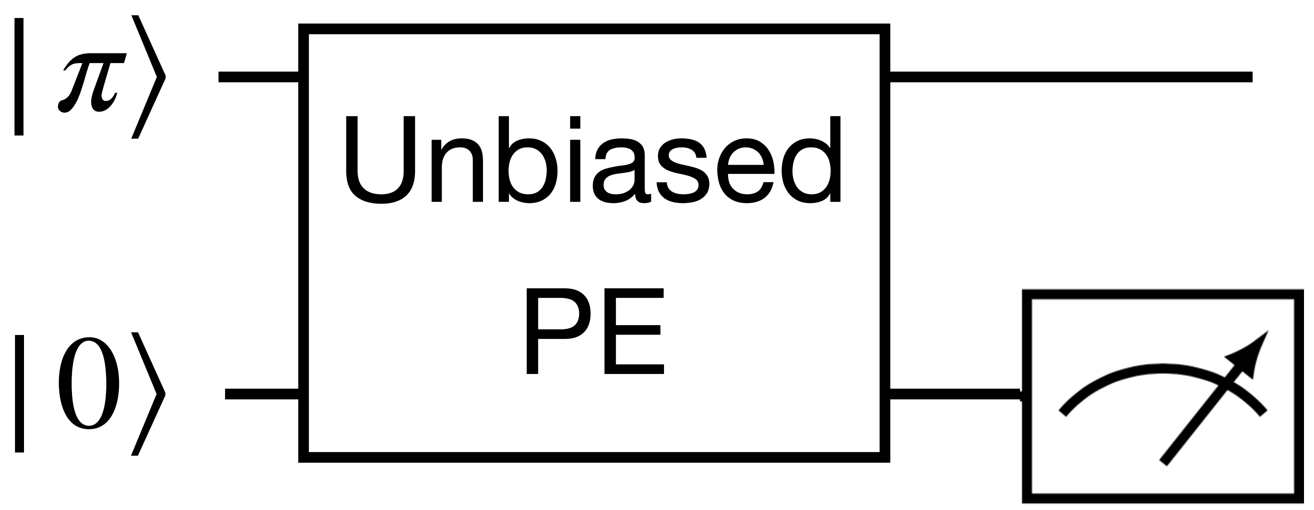
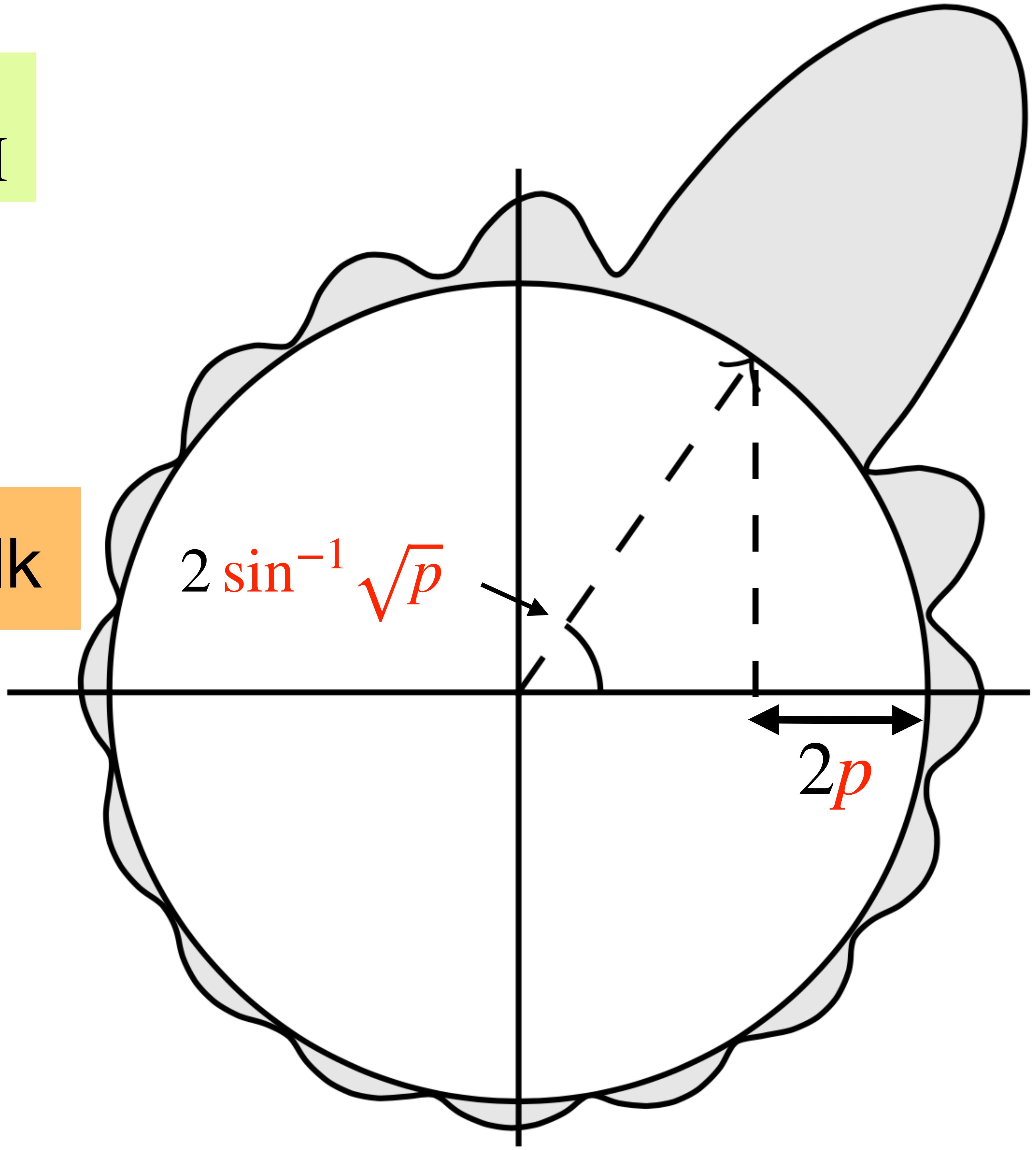


Unknown

$|\pi_-\rangle + e^{i\gamma}|\pi_+\rangle$

Eigenvectors of U



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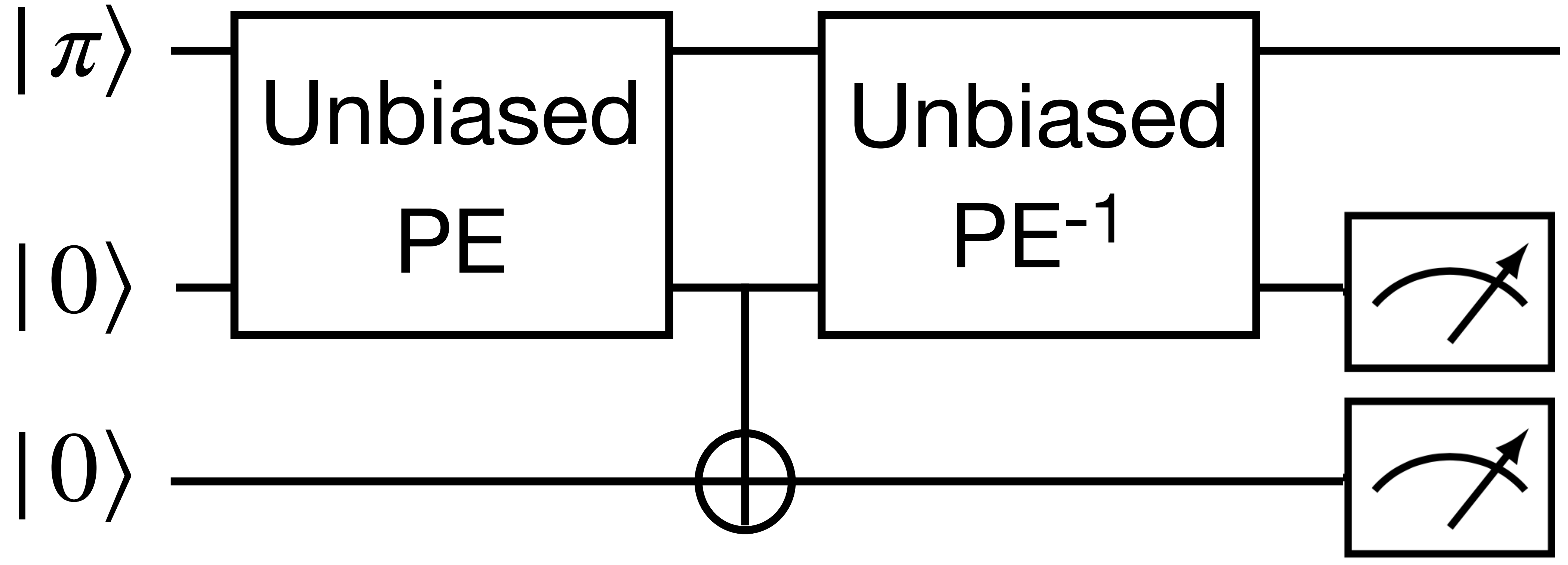
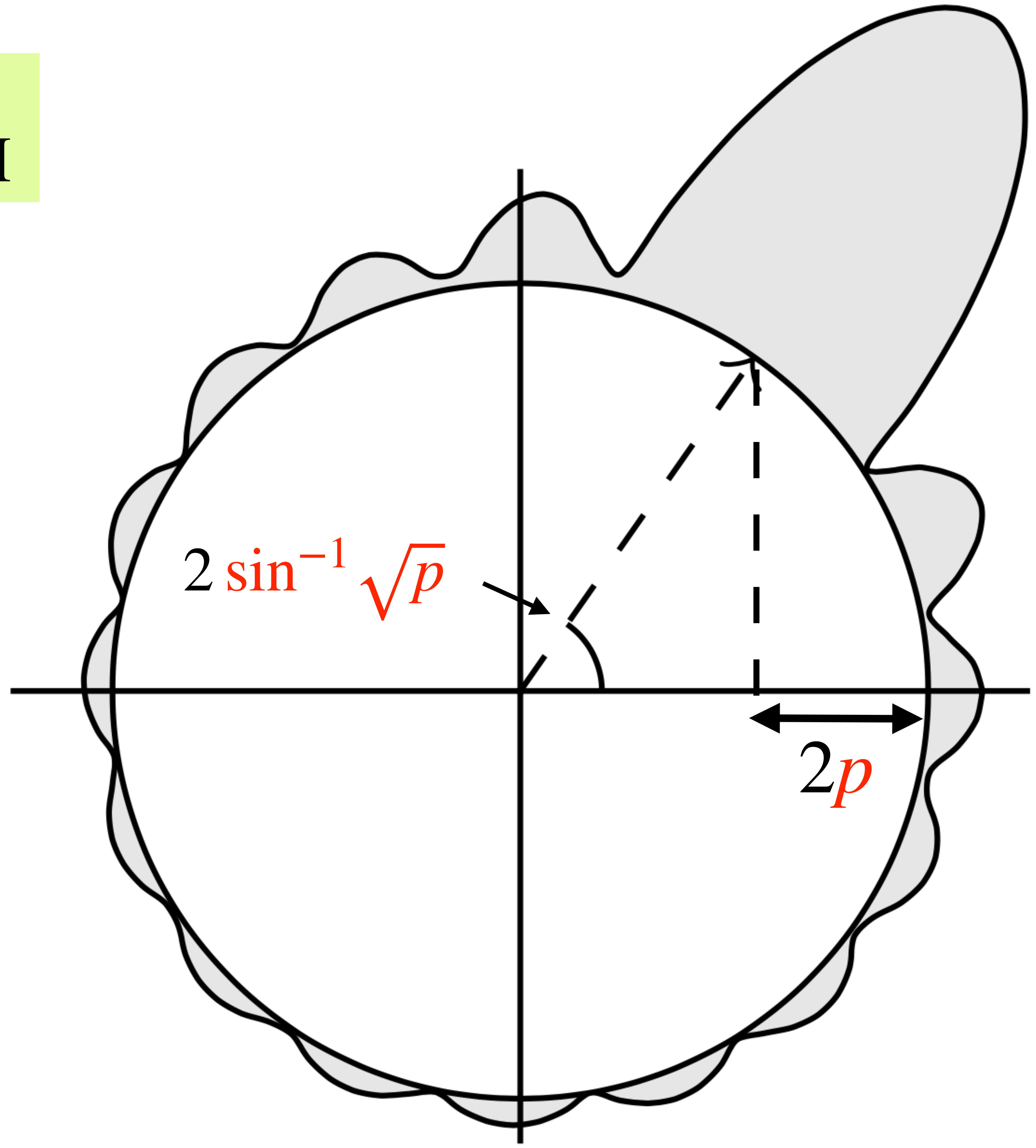
Restoring $|\pi\rangle$? Rewind the estimation





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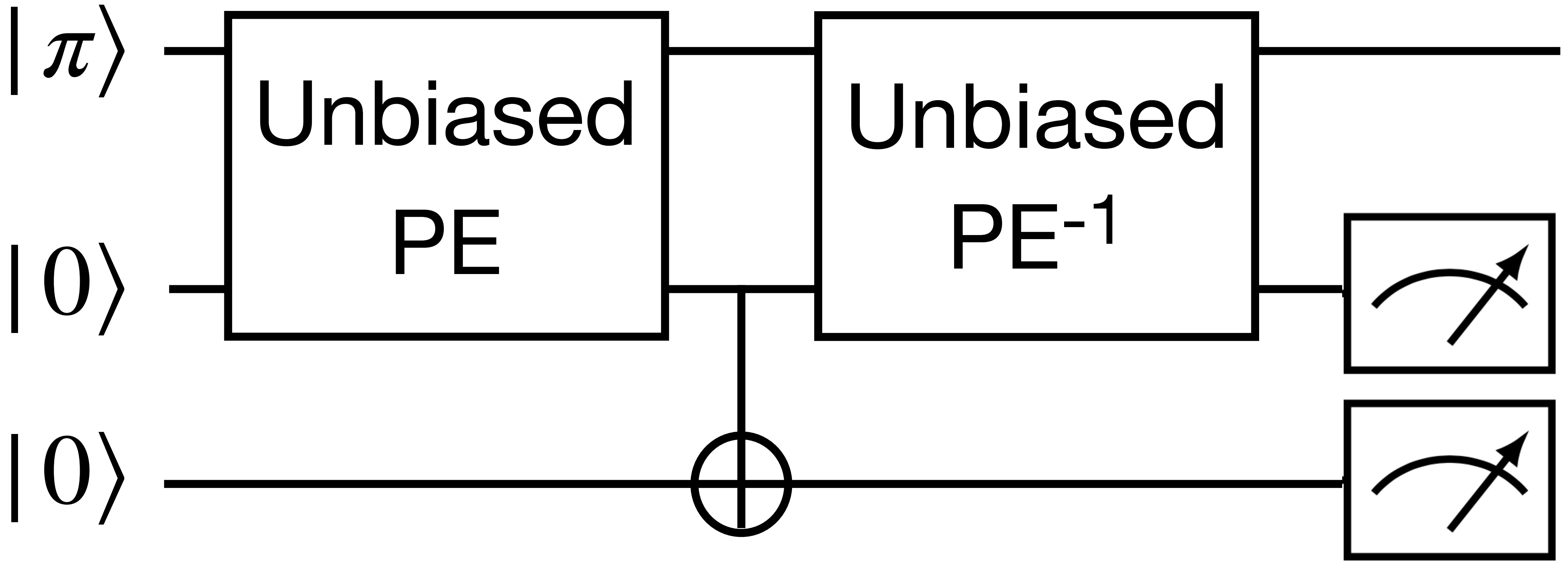
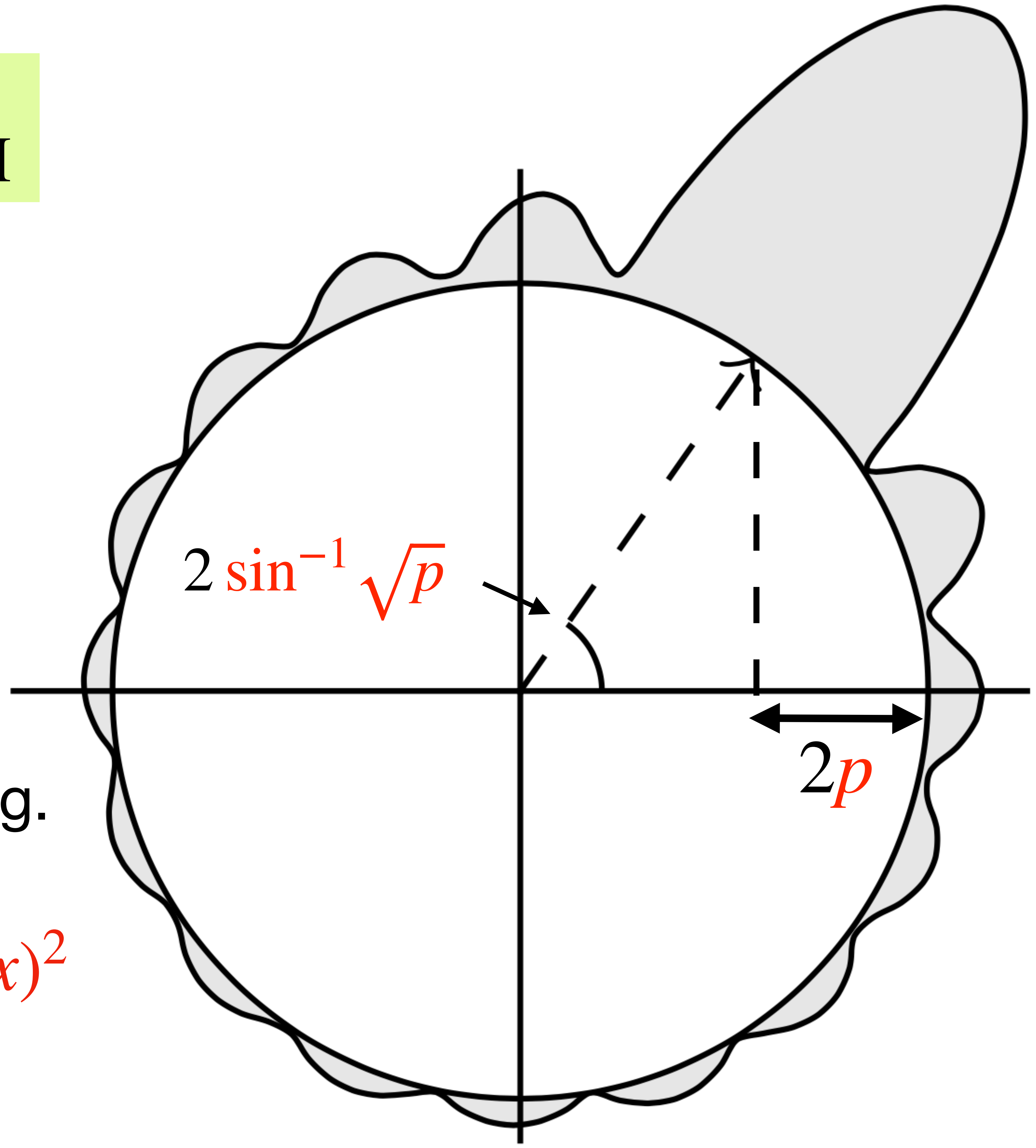


$|\pi\rangle$ if 0 in 2nd reg.

$\Pr(0) = \sum_x d(x)^2$

Density $d(x)$



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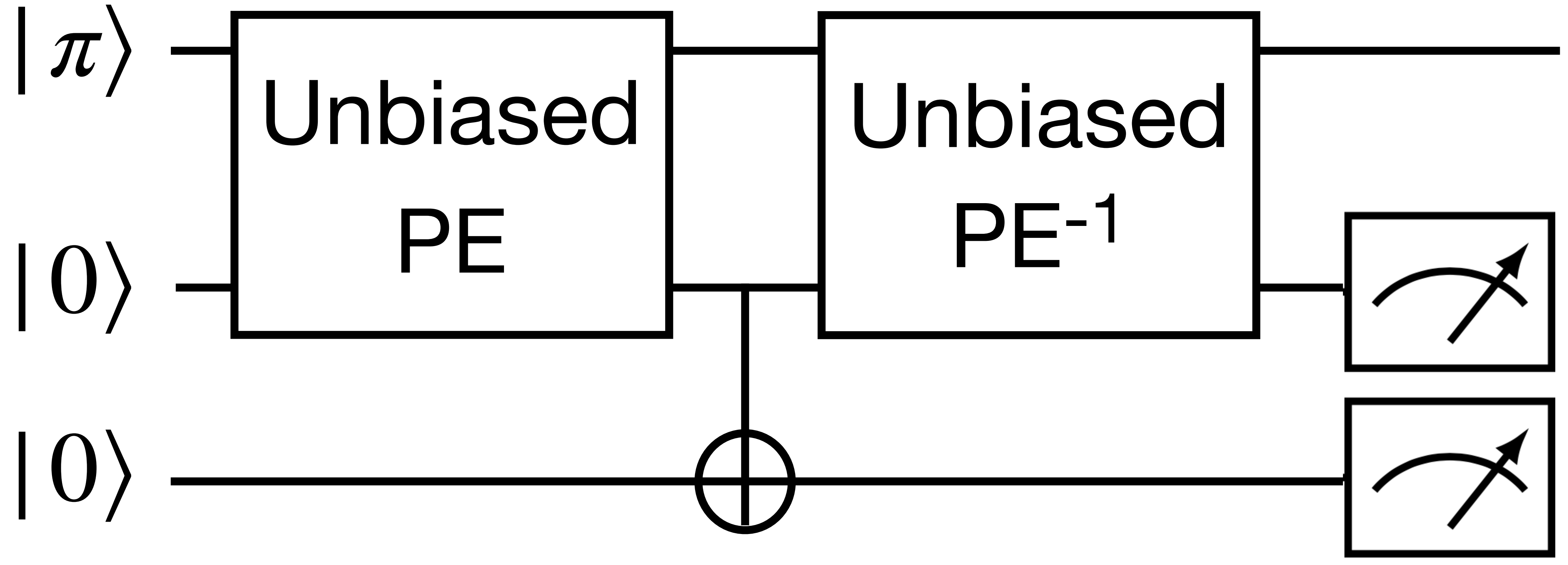

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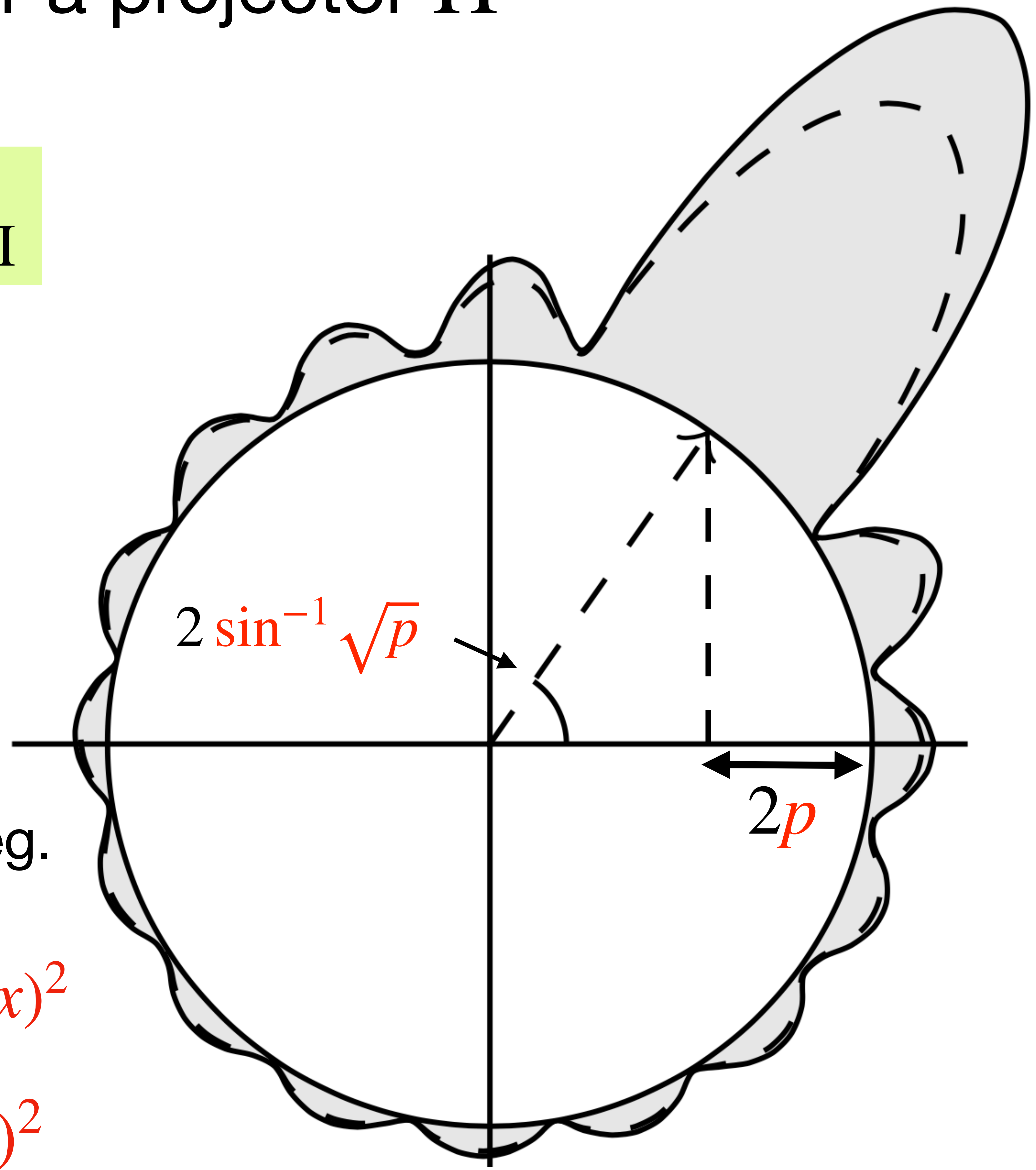
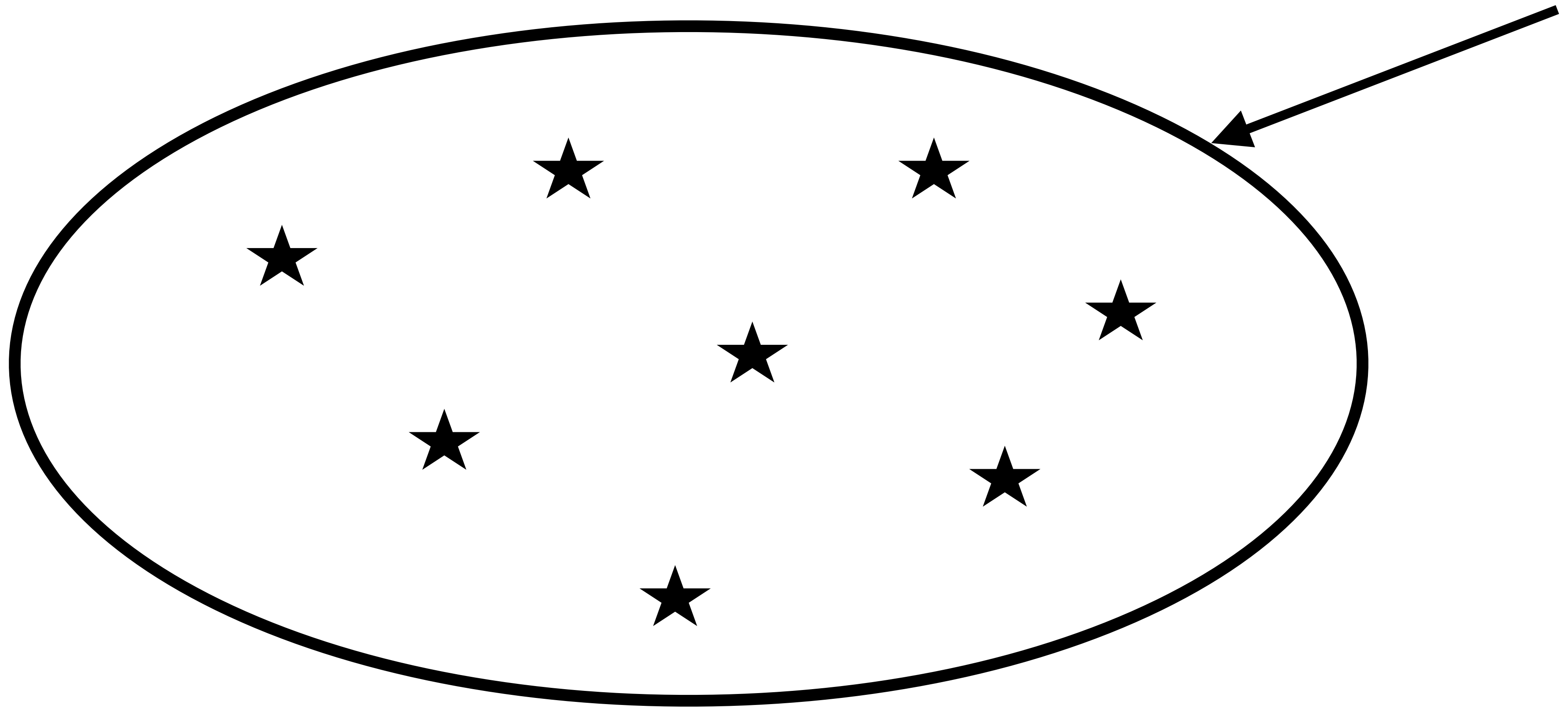
$\Pr(0) = \sum_x d(x)^2$

Density $\propto d(x)^2$



$\Omega =$



Examples:

- independent sets
- k-colorings
- matchings
- (volume of convex body)
- (Ising model)
- ...

Approximate the size $|\Omega|$ in time

$$\approx \log^{3/4} |\Omega| \times \sqrt{\text{classical mixing time}}$$

Previous work:

$$\log |\Omega| \times \ldots$$

Open question: $\log^{1/2} |\Omega| \times \sqrt{\text{class. mixing time}}$?