Quantum Chebyshev's Inequality and Applications

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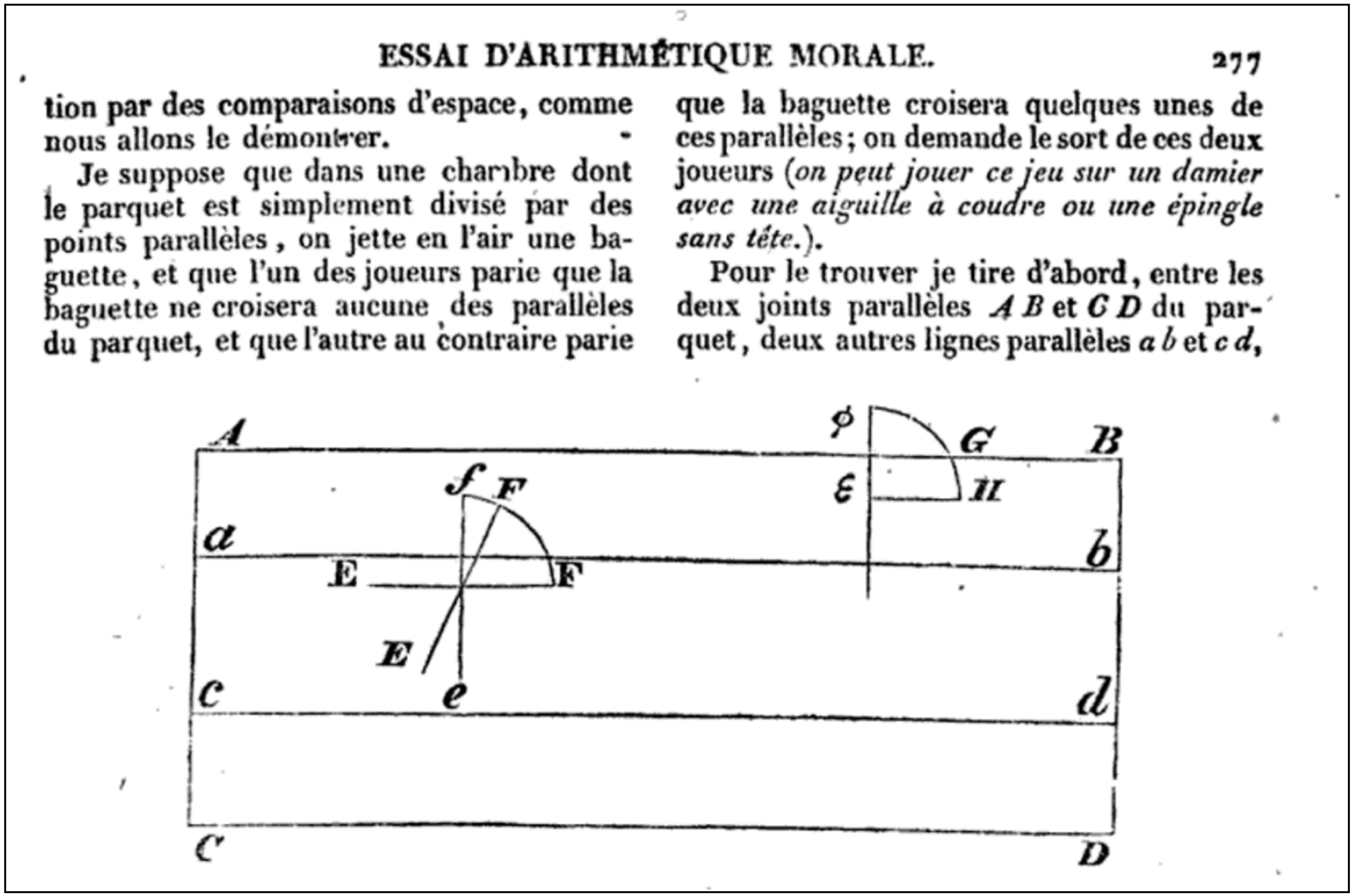
Buffon’s needle

A needle dropped randomly on a floor with equally spaced parallel lines will cross one of the lines with probability **2/π**.



tion par des comparaisons d'espace, comme nous allons le démontrer.

Je suppose que dans une chambre dont le parquet est simplement divisé par des points parallèles, on jette en l'air une baguette, et que l'un des joueurs parie que la baguette ne croisera aucune des parallèles du parquet, et que l'autre au contraire parie que la baguette croisera quelques unes de ces parallèles; on demande le sort de ces deux joueurs (*on peut jouer ce jeu sur un damier avec une aiguille à coudre ou une épingle sans tête*).

Pour le trouver je tire d'abord, entre les deux joints parallèles *A B* et *C D* du parquet, deux autres lignes parallèles *a b* et *c d*,

Buffon, G., *Essai d'arithmétique morale*, 1777.

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Law of large numbers: $$\frac{x_1 + \ldots + x_n}{n} \xrightarrow{n \to \infty} \mathbf{E}(X)$$

Empirical mean: $\widetilde{\mu} = \frac{x_1 + \ldots + x_n}{n}$ with $x_1, \ldots, x_n \sim X$

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Chebyshev's Inequality:

multiplicative error 0 < ε < 1

Objective: $|\widetilde{\mu} - \mathbf{E}(X)| \leq \epsilon \mathbf{E}(X)$ with high probability ($\mathbf{E}(X), \mathbf{Var}(X) \neq 0$ finite)

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In practice: given an upper-bound $\Delta^2 \geq \dfrac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2}$, take $n = \Omega\left(\dfrac{\Delta^2}{\epsilon^2}\right)$ samples

Counting with Markov chain Monte Carlo methods:

Counting vs. sampling [Jerrum, Sinclair'96] [Štefankovič et al.'09], Volume of convex bodies [Dyer, Frieze'91], Permanent [Jerrum, Sinclair, Vigoda'04]

Data stream model:

Frequency moments, Collision probability [Alon, Matias, Szegedy'99] [Monemizadeh, Woodruff'] [Andoni et al.'11] [Crouch et al.'16]

Testing properties of distributions:

Closeness [Goldreich, Ron'11] [Batu et al.'13] [Chan et al.'14], Conditional independence [Canonne et al.'18]

Estimating graph parameters:

Number of connected components, Minimum spanning tree weight [Chazelle, Rubinfeld, Trevisan'05], Average distance [Goldreich, Ron'08], Number of triangles [Eden et al. 17]

etc.

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Quantum sample: one (controlled-)**execution** of a **quantum sampler** S_X or S_X^{-1}, where

$$S_X|0\rangle = \sum_{x\in\Omega} \sqrt{p_x}\,|\psi_x\rangle\,|x\rangle$$

with $\psi_x =$ arbitrary unit vector

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[Montanaro'15]	$\frac{\Delta^{\mathbf{2}}}{\epsilon}$	$\Delta^2 \geq \frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2}$
[Li, Wu'17]	$\frac{\Delta}{\epsilon} \cdot \frac{H}{L}$	$\Delta^2 \geq \frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2}$ **L** ≤ E(X) ≤ **H**

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[Li, Wu'17]	$\frac{\Delta}{\epsilon} \cdot \frac{H}{L}$	$\Delta^2 \geq \frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2}$	L ≤ E(X) ≤ H
Our result	$\frac{\Delta}{\epsilon} \cdot \log^3\left(\frac{H}{\mathbf{E}(X)}\right)$	$\Delta^2 \geq \frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2}$	E(X) ≤ H

Our Approach

Amplitude Estimation Algorithm [Brassard et al.'02] [Brassard et al.'11] [Wocjan et al.'09] [Montanaro'15]

Input: Random variable **X** on sample space **Ω** ⊂ **[0,B]**

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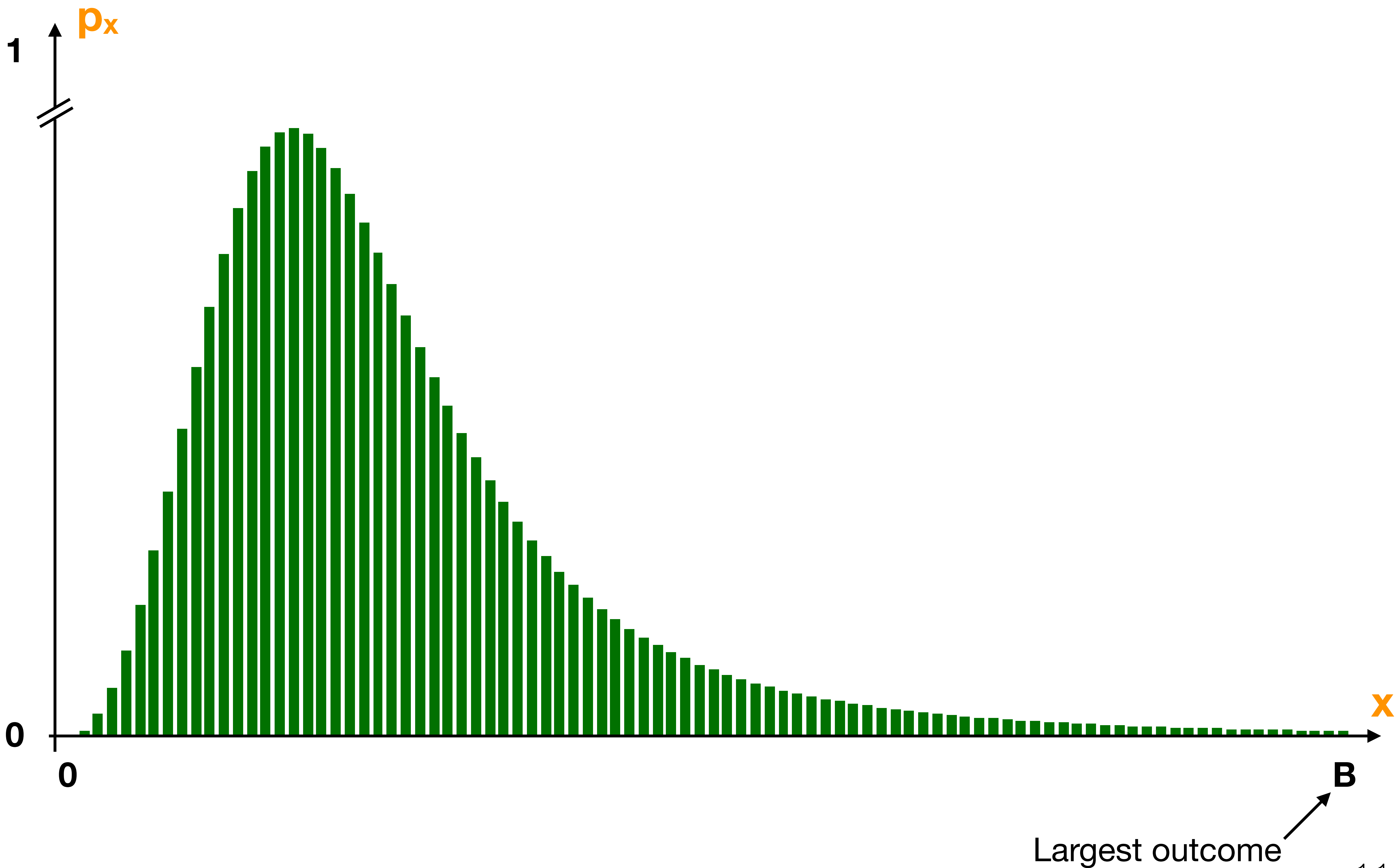
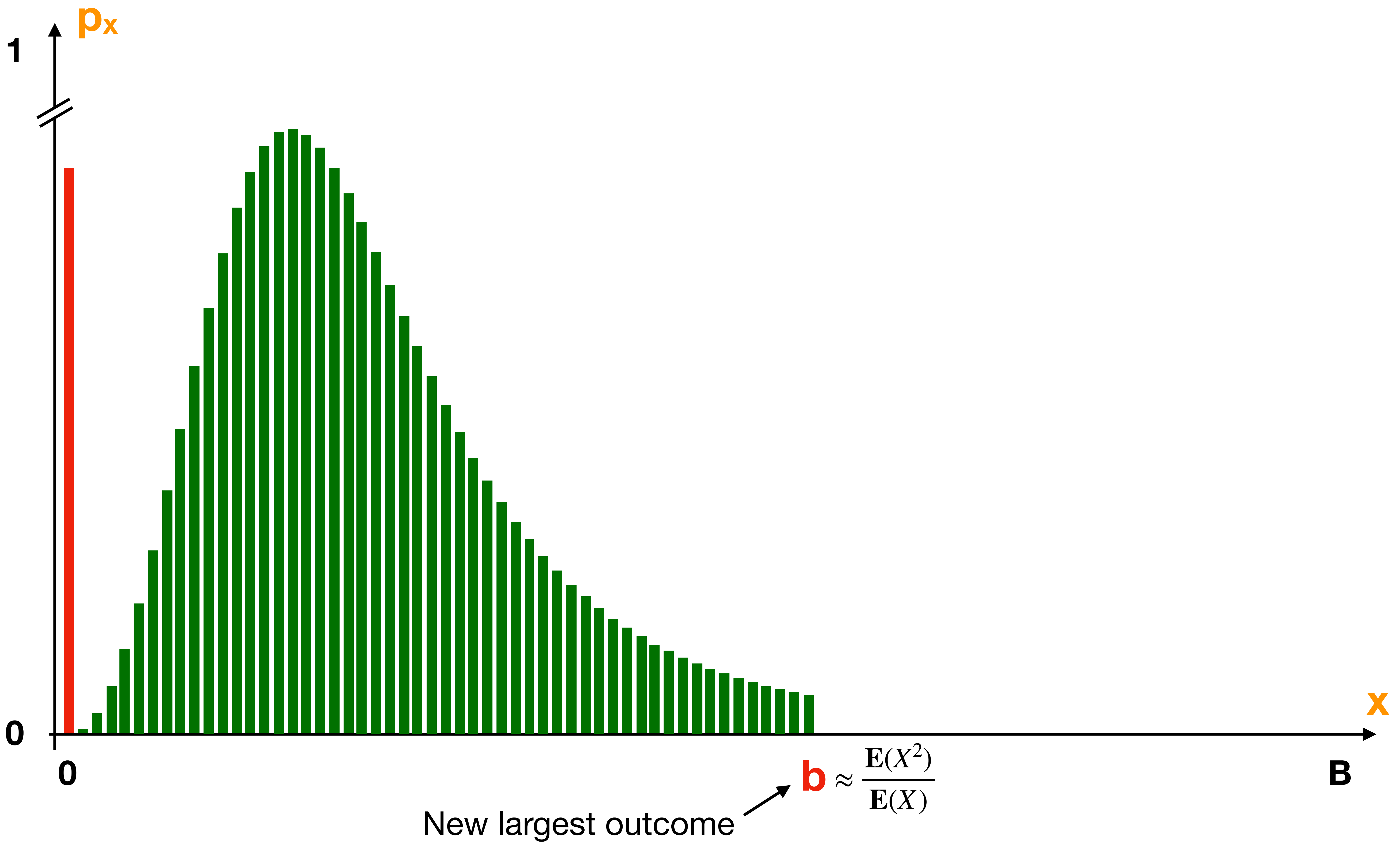
If $B \gg \frac{\mathbf{E}(X^2)}{\mathbf{E}(X)}$?

Random variable X



Random variable X_b



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Lemma: If $b \geq \frac{\mathbf{E}(X^2)}{\epsilon\mathbf{E}(X)}$ then $(1-\epsilon)\mathbf{E}(X) \leq \mathbf{E}(X_b) \leq \mathbf{E}(X)$.

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[Brassard et al.'02]

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$$1/\Delta^2$$

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Ingredient 3: **If** $b \approx \mathbf{E}(X) \cdot \Delta^2$ **then** $\dfrac{\mathbf{E}(X_b)}{b} \overset{\star}{\approx} \dfrac{\mathbf{E}(X)}{b} \approx \dfrac{1}{\Delta^2}$

Applications

Application 1: approximating graph parameters

Input: graph G=(V,E) with **n** vertices, **m** edges, **t** triangles

Query access: unitaries $O_{\mathrm{deg}}|v\rangle|0\rangle = |v\rangle|\mathrm{deg}(v)\rangle$ *(degree query)*

$O_{\mathrm{pair}}|v\rangle|w\rangle|0\rangle = |v\rangle|w\rangle|(v,w) \in E\ ?\rangle$ *(pair query)*

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Result: $\widetilde{\Theta}\left(\frac{\sqrt{n}}{m^{1/4}}\right)$ degree/neighbor quantum queries to approximate **m**

$\widetilde{\Theta}\left(\frac{\sqrt{n}}{t^{1/6}} + \frac{m^{3/4}}{\sqrt{t}}\right)$ degree/pair/neighbor quantum queries to approximate **t**

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[Goldreich, Ron'08] [Seshadhri'15]

$\widetilde{\Theta}\left(\frac{\sqrt{n}}{t^{1/6}} + \frac{m^{3/4}}{\sqrt{t}}\right)$ degree/pair/neighbor quantum queries to approximate **t**

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[Eden, Levi, Ron'15] [Eden, Levi, Ron, Seshadhri'17]

Application 2: frequency moments in the streaming model

Input: (finite) stream of updates $\mathbf{x_i} \leftarrow \mathbf{x_i} + \delta$ on **x** = (0,…,0) of **dimension n**

Output: (at the end of the stream) approximate of $F_k = \sum_{i=1}^{n} |x_i|^k$ (moment of order k ≥ 3)

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Result: $\mathbf{M = \widetilde{O}\left(\frac{n^{1-2/k}}{P^2}\right)}$ qubits of memory

(vs. $M = \widetilde{\Theta}\left(\frac{n^{1-2/k}}{P}\right)$ classical bits of memory)

[Monemizadeh, Woodruff'10]
[Andoni, Krauthgamer, Onak'11]

Conclusion

The **mean** of a random variable X can be estimated with **multiplicative error ε** using $\widetilde{O}\left(\frac{\Delta}{\epsilon} \cdot \log^3\left(\frac{H}{E(X)}\right)\right)$ **quantum samples**, given $\Delta^2 \geq \frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2}$ and $H \geq \mathbf{E}(X)$.

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Lower bound: $\Omega\left(\frac{\Delta - 1}{\epsilon}\right)$ quantum samples

or $\Omega\left(\frac{\Delta^2 - 1}{\epsilon^2}\right)$ copies of the state $S_X|0\rangle = \sum_{x\in\Omega} \sqrt{p_x}\,|\psi_x\rangle|x\rangle$

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Extra slides

Subroutine: the Amplitude Estimation algorithm

Sampler: $S_X|0\rangle = \sum_{x\in\Omega} \sqrt{p_x}\,|\psi_x\rangle\,|x\rangle$ on sample space **$\Omega \subset$ [0,B]**

Result: $O\left(\frac{\sqrt{B}}{\epsilon\sqrt{\mathbf{E}(X)}}\right)$ quantum samples to obtain $|\tilde{\mu} - \mathbf{E}(X)| \leq \epsilon\mathbf{E}(X)$

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Reduction to a Bernoulli sampler [Brassard et al.'11] [Wocjan et al.'09] [Montanaro'15]:

$$\sum_{x\in\Omega} \sqrt{p_x}\,|\psi_x\rangle|x\rangle|0\rangle \xrightarrow{\text{Controlled rotation}} \sum_{x\in\Omega} \sqrt{p_x}\,|\psi_x\rangle|x\rangle\left(\sqrt{1-\frac{x}{B}}\,|0\rangle + \sqrt{\frac{x}{B}}\,|1\rangle\right)$$

$$\xrightarrow{\text{Reordering}} \sqrt{1-\frac{\mathbf{E(X)}}{\mathbf{B}}}\,|\varphi_0\rangle|\mathbf{0}\rangle + \sqrt{\frac{\mathbf{E(X)}}{\mathbf{B}}}\,|\varphi_1\rangle|\mathbf{1}\rangle = S_Y|0\rangle$$

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Expectation of a Bernoulli sampler [Brassard et al.'02]:

$$\mathbf{S_Y}|\mathbf{0}\rangle = \sqrt{\mathbf{1} - \frac{\mathbf{E(X)}}{\mathbf{B}}}\,|\varphi_0\rangle|\mathbf{0}\rangle + \sqrt{\frac{\mathbf{E(X)}}{\mathbf{B}}}\,|\varphi_1\rangle|\mathbf{1}\rangle$$

Subroutine: the Amplitude Estimation algorithm

Sampler: $S_X|0\rangle = \sum_{x\in\Omega} \sqrt{p_x}\,|\psi_x\rangle|x\rangle$ on sample space **Ω ⊂ [0,B]**

Result: $O\left(\frac{\sqrt{B}}{\epsilon\sqrt{\mathbf{E}(X)}}\right)$ quantum samples to obtain $|\widetilde{\mu} - \mathbf{E}(X)| \leq \epsilon\mathbf{E}(X)$

Expectation of a Bernoulli sampler [Brassard et al.'02]:

$$\mathbf{S_Y}|\mathbf{0}\rangle = \sqrt{\mathbf{1} - \frac{\mathbf{E(X)}}{\mathbf{B}}}\,|\varphi_0\rangle|\mathbf{0}\rangle + \sqrt{\frac{\mathbf{E(X)}}{\mathbf{B}}}\,|\varphi_1\rangle|\mathbf{1}\rangle$$

Step 0: the Grover's operator $\mathbf{G} = \mathbf{S_Y^{-1}}(I - 2|0\rangle\langle 0|)\mathbf{S_Y}(I - 2I\otimes|1\rangle\langle 1|)$ has eigenvalues $e^{\pm 2i\theta}$, where $\theta = \sin^{-1}(\sqrt{\mathbf{E}(X)/B})$.

Step 1: use the Phase Estimation Algorithm on G for $t \geq \Omega(\sqrt{B}/(\epsilon\sqrt{\mathbf{E}(X)}))$ steps (i.e. using t quantum samples), to get an estimate $\widetilde{\theta}$ of $\pm\theta$.

Step 2: output $\sin^2(\widetilde{\theta})$ as an estimate to E(X)/B. $\quad(\widetilde{\mu} = B\cdot\sin^2(\widetilde{\theta}))$

No a priori information on $E(X^2)/E(X)^2$

Result: There is an **optimal** algorithm that approximates the mean of any quantum sampler S_X over $\Omega \subset [0,B]$ with

$$\widetilde{\Theta}\left(\frac{\sqrt{B}}{\sqrt{\epsilon \mathbf{E}(X)}} + \frac{\mathbf{E}(X^2)}{\epsilon \mathbf{E}(X)}\right)$$

quantum samples, when there is no a priori information on X.

→ Quantization of [Dagum, Karp, Luby, Ross'00]

Lemma: If $b \geq \frac{\mathbf{E}(X^2)}{\epsilon \mathbf{E}(X)}$ then $(1-\epsilon)\mathbf{E}(X) \leq \mathbf{E}(X_{<b}) \leq \mathbf{E}(X)$.

Lemma: If $b \geq 10^4 \cdot \mathbf{E}(X)\Delta^2$ then $\frac{\mathbf{E}(X_{<b})}{b} \leq \frac{1}{10^4 \cdot \Delta^2}$

Lemma: If $b \geq \dfrac{\mathbf{E}(X^2)}{\epsilon \mathbf{E}(X)}$ then $(1-\epsilon)\mathbf{E}(X) \leq \mathbf{E}(X_{<b}) \leq \mathbf{E}(X)$.

Proof:

- $\mathbf{E}(X_{\geq b}) \leq \dfrac{\mathbf{E}(X^2)}{b} \leq \epsilon \mathbf{E}(X)$
- $\mathbf{E}(X_{<b}) = \mathbf{E}(X) - \mathbf{E}(X_{\geq b}) \geq (1-\epsilon)\mathbf{E}(X)$

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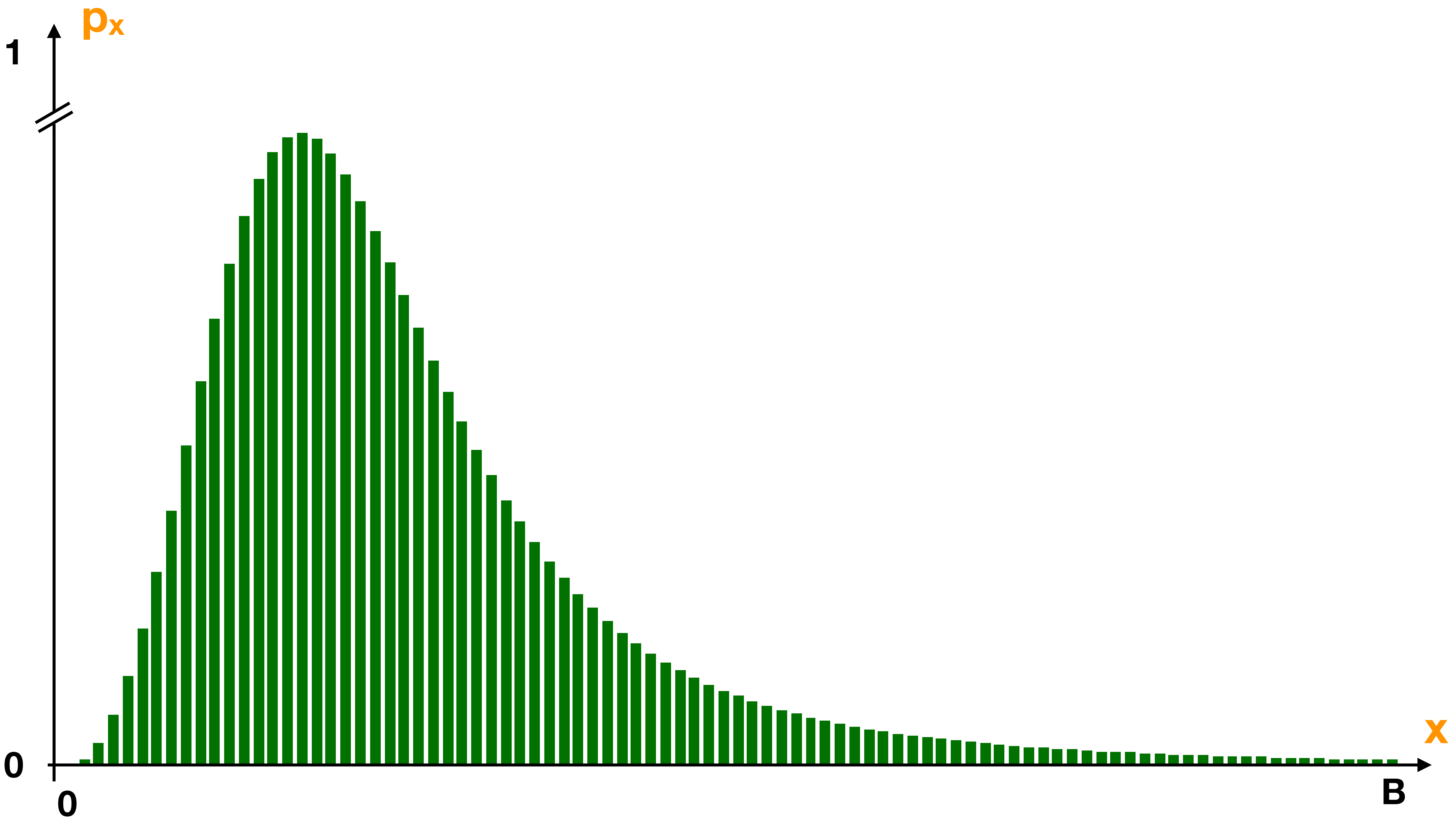
Proof: $\dfrac{\mathbf{E}(X_{<b})}{b} \leq \dfrac{\mathbf{E}(X)}{10^4 \mathbf{E}(X)\Delta^2} \leq \dfrac{1}{10^4 \cdot \Delta^2}$

Example



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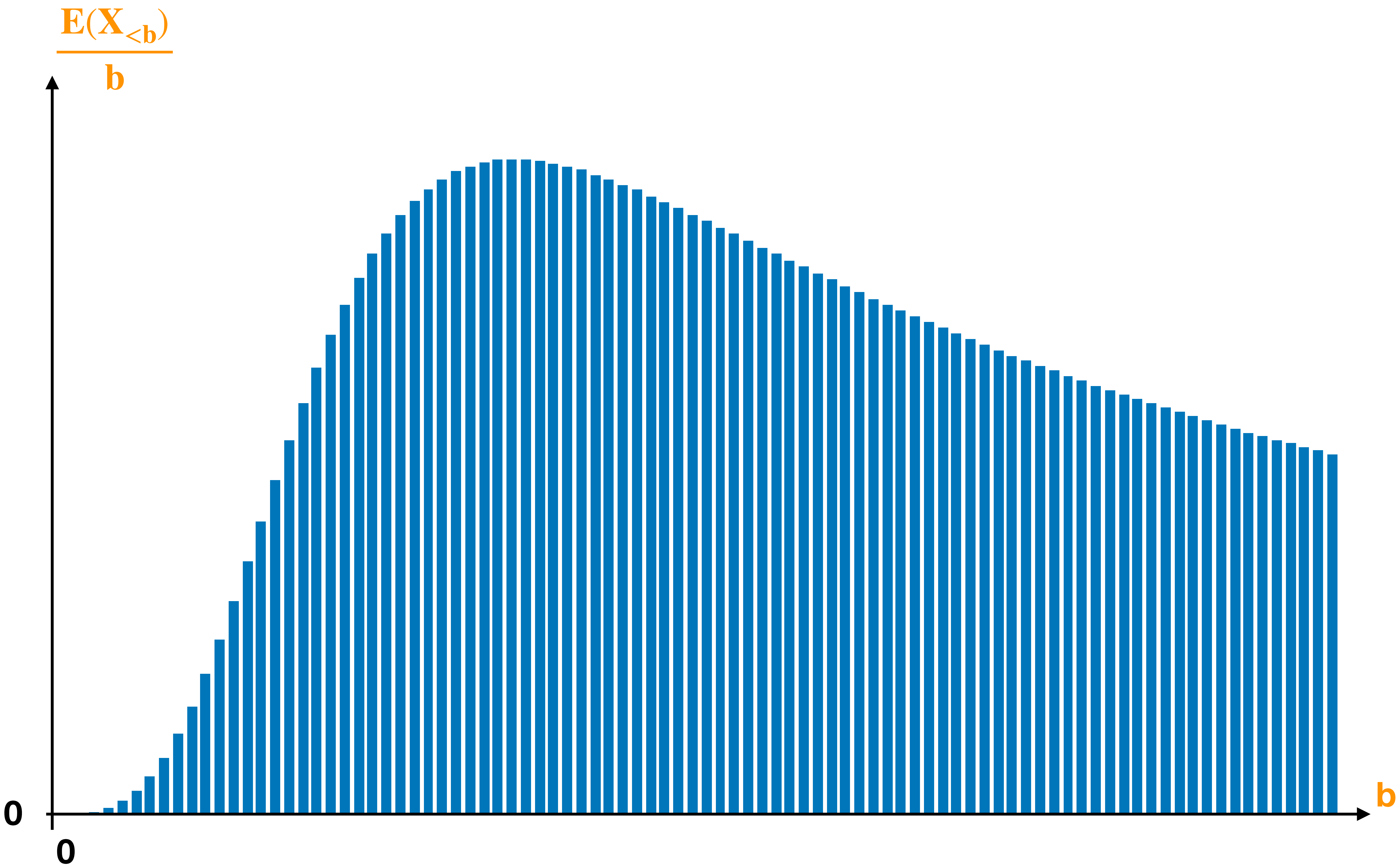

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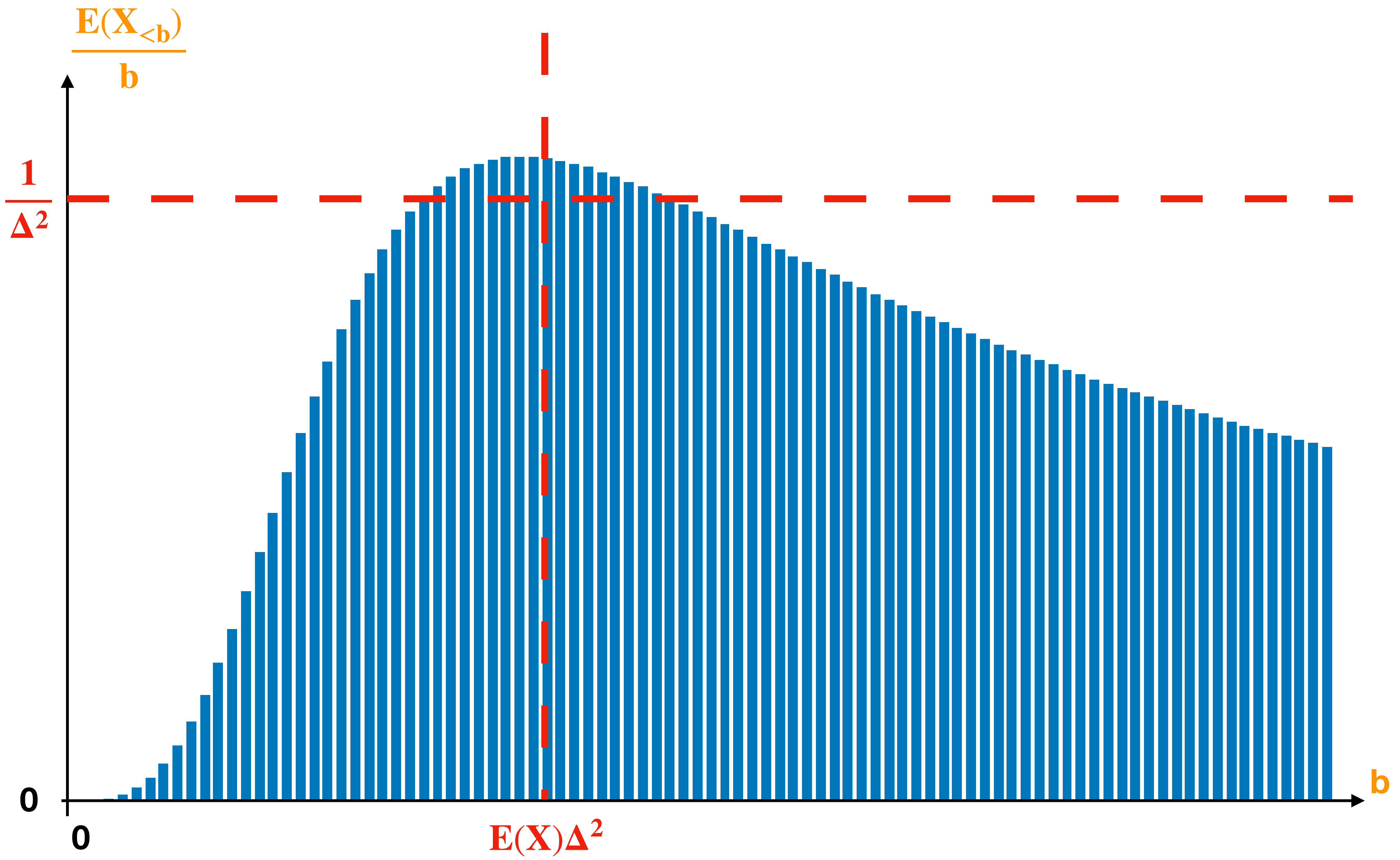

Final algorithm:

Step 1: Logarithmic search on b until **Amplitude-Estimation**$(S_{X_{<b}}, \Delta) \neq 0$

⟶ get $2 \cdot \mathbf{E}(X)\Delta^2 \leq b \leq 10^4 \cdot \mathbf{E}(X)\Delta^2$ with high probability

$$\Delta \cdot \log^3\left(\frac{H}{\mathbf{E}(X)}\right)$$

Step 2: Set threshold $d = b/\epsilon$ and output **Amplitude-Estimation**$(S_{X_{<d}}, \Delta/\epsilon^{3/2}) \neq 0$

⟶ get $|\widetilde{\mu} - \mathbf{E}(X)| \leq \epsilon\mathbf{E}(X)$ with high probability

$$\Delta/\epsilon^{3/2}$$

Step 2bis: Slightly refined algorithm, adapted from [Heinrich'01, Montanaro'15]

$$\Delta/\epsilon$$

Application 1: counting the number of edges in a graph

Estimator X :=

1. Sample a vertex $v \in V$ uniformly at random
2. Sample a neighbor w of v uniformly at random
3. If $\deg(v) < \deg(w)$ (or $\deg(v) = \deg(w)$ and $v <_{lex} w$)

   Output $\mathbf{n*deg(v)}$

   Else

   Output $\mathbf{0}$

λ(v,w)

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Lemma: $E(X) = m$ and $E(X^2)/E(X)^2 \leq O(\sqrt{n})$. (when $m \geq \Omega(n)$)

[Goldreich, Ron'08]
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$$S_X|0\rangle = \sum_{v \in V} \sum_{w \in N(v)} \frac{1}{\sqrt{n \cdot \deg(v)}} |v\rangle |w\rangle |\lambda(v,w)\rangle$$

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Result: $O(n^{1/4}/\varepsilon)$ quantum samples (= quantum queries) to approximate m.

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Result: $\Theta(n^{1/2}/m^{1/4})$ quantum samples (= quantum queries) to approximate m.

Application 2: frequency moments in the streaming model

$$\mathbf{x} = \begin{array}{cccccccc} 1 & 2 & 3 & & & & & n \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \cdots & \boxed{0} & \boxed{0} & \boxed{0} \end{array}$$

Stream of **updates** to x:

Application 2: frequency moments in the streaming model

	1	2	3					n
x =	**0**	**0**	**5**	**0**	…	**0**	**0**	**0**

Stream of **updates** to x: **(3,+5)**

Application 2: frequency moments in the streaming model

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Stream of **updates** to x: **(3,+5) ; (2,-6)**

Application 2: frequency moments in the streaming model

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1 pass + memory $\mathbf{M = \frac{n^{1-2/k}}{P}}$

||

1 sample from a random variable X with

$\mathbf{E(X) \approx F_k}$ and $\mathbf{E(X^2)/E(X)^2 \leq P \cdot F_k^2}$

[Monemizadeh, Woodruff'10]
[Andoni, Krauthgamer, Onak'11]

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Quantumly: $P^2M = O(n^{1-2/k})$

1 pass + memory $\mathbf{M = \frac{n^{1-2/k}}{P^2}}$

||

1 **quantum** sample* S_X from a r.v. X with

$\mathbf{E(X) \approx F_k}$ and $\mathbf{E(X^2)/E(X)^2 \leq (P \cdot F_k)^2}$

* $\mathbf{S_X^{-1}}$ can be done in one pass also

Application 3: counting the number of triangles in a graph

More complicated than edges… [Eden, Levi, Ron'15] [Eden, Levi, Ron, Seshadhri'17]

Main subroutine: estimator X for the number of triangles adjacent to any vertex v

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Variable-time Amplitude Estimation: estimate the amplitude when some "branches" of the computation stop earlier than the others

Result:

$$\widetilde{\Theta}\left(\frac{\sqrt{n}}{t^{1/6}} + \frac{m^{3/4}}{\sqrt{t}}\right)$$ quantum queries for triangle counting

vs. $$\widetilde{\Theta}\left(\frac{n}{t^{1/3}} + \frac{m^{3/2}}{t}\right)$$ classical queries