Quantum Chebyshev's Inequality and Applications

Yassine Hamoudi, Frédéric Magniez

IRIF, Université de Paris, CNRS

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In practice: we often know $\Delta^2 \ge \dfrac{\mathrm{E}(X^2)}{\mathrm{E}(X)^2} = \dfrac{\mathrm{Var}(X^2)}{\mathrm{E}(X)^2} + 1$ → take $\dfrac{\Delta^2}{\epsilon^2}$ samples

Counting with Markov chain Monte Carlo methods:

Counting vs. sampling [Jerrum, Sinclair'96] [Štefankovič et al.'09], Volume of convex bodies [Dyer, Frieze'91], Permanent [Jerrum, Sinclair, Vigoda'04]

Data stream model:

Frequency moments, Collision probability [Alon, Matias, Szegedy'99] [Monemizadeh, Woodruff'] [Andoni et al.'11] [Crouch et al.'16]

Testing properties of distributions:

Closeness [Goldreich, Ron'11] [Batu et al.'13] [Chan et al.'14], Conditional independence [Canonne et al.'18]

Estimating graph parameters:

Number of connected components, Minimum spanning tree weight [Chazelle, Rubinfeld, Trevisan'05], Average distance [Goldreich, Ron'08], Number of triangles [Eden et al. 17]

etc.

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Question: can we estimate E(X) with less samples in the quantum setting?

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(Chebyshev)	$\frac{\Delta^2}{\epsilon^2}$ given $\Delta^2 \geq \frac{\mathrm{E}(X^2)}{\mathrm{E}(X)^2}$	

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Our Approach

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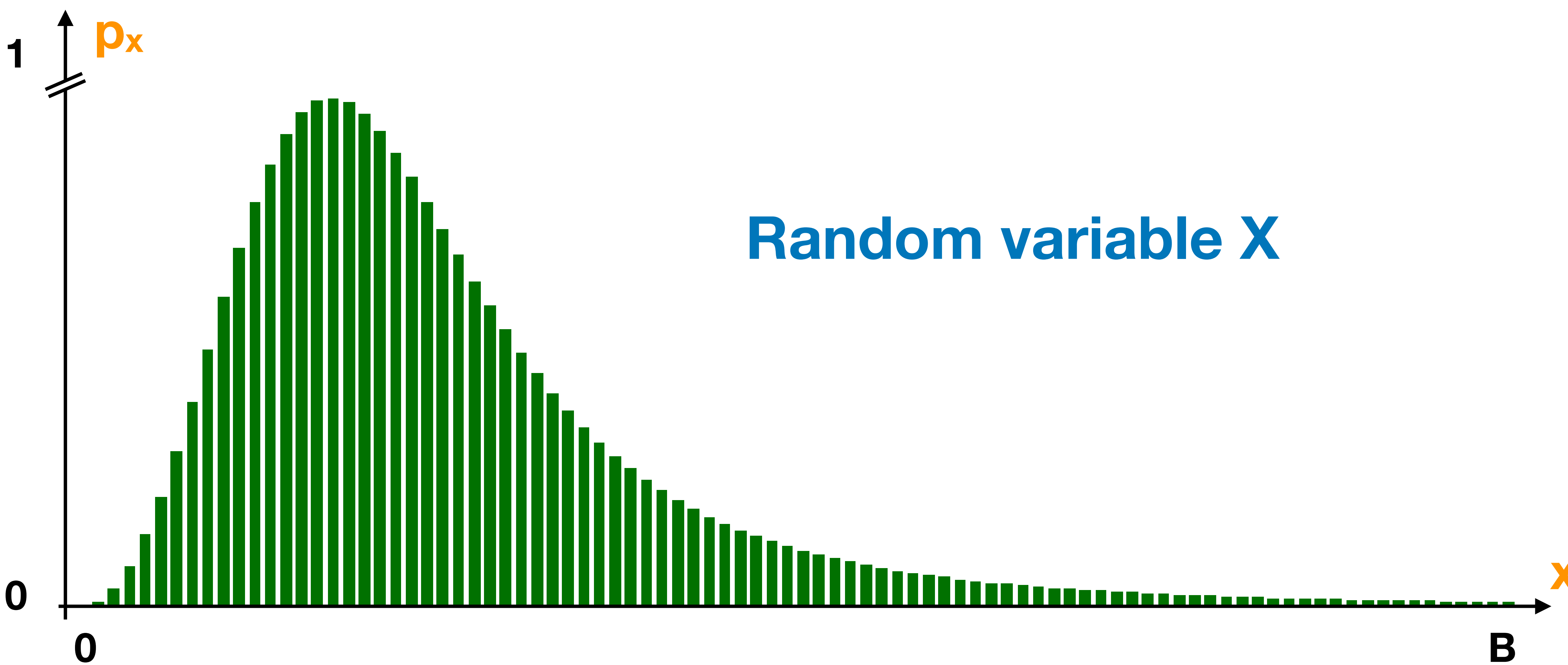

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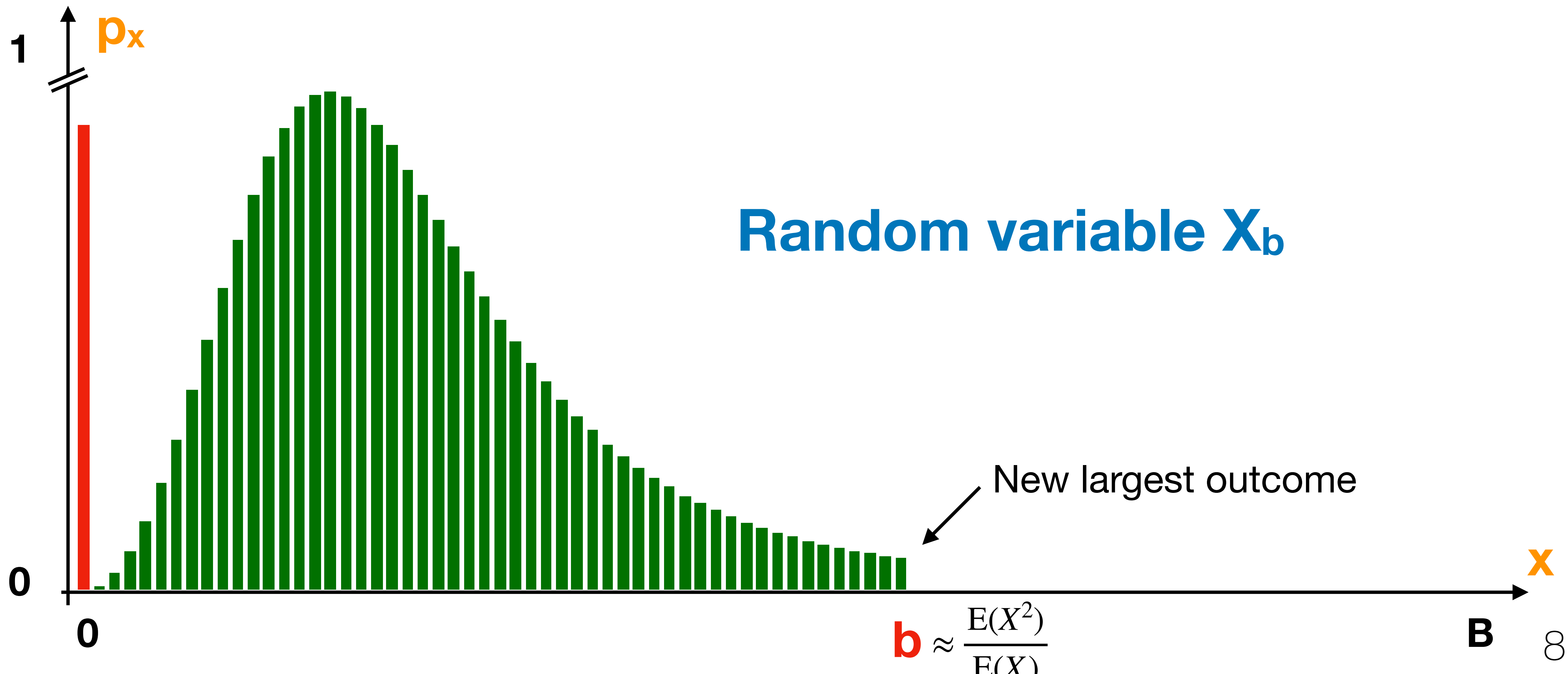

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Lemma: If $b \geq \frac{\mathrm{E}(X^2)}{\epsilon\mathrm{E}(X)}$ then $(1-\epsilon)\mathrm{E}(X) \leq \mathrm{E}(X_b) \leq \mathrm{E}(X)$.

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Problem: $\frac{\mathrm{E}(X^2)}{\mathrm{E}(X)}$ is unknown… but we know $\Delta^2 \geq \frac{\mathrm{E}(X^2)}{\mathrm{E}(X)^2}$ $\longrightarrow$ $b \approx \mathrm{E}(X) \cdot \Delta^2$?

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Theorem: the first non-zero $\widetilde{\mu}_i$ is obtained w.h.p. when:

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Applications

Application 1: approximating graph parameters

Input: graph G=(V,E) with **n** vertices, **m** edges, **t** triangles

Query access: unitaries

$$O_{\mathrm{deg}}|v\rangle|0\rangle = |v\rangle|\mathrm{deg}(v)\rangle$$ *(degree query)*

$$O_{\mathrm{pair}}|v\rangle|w\rangle|0\rangle = |v\rangle|w\rangle|(v,w) \in E\ ?\rangle$$ *(pair query)*

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[Goldreich, Ron'08] [Seshadhri'15]

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[Eden, Levi, Ron'15] [Eden, Levi, Ron, Seshadhri'17]

Application 2: frequency moments in the streaming model

Initially: **x** = (0,…,0) of **dimension n**

Input: stream of updates $\mathbf{x_i} \leftarrow \mathbf{x_i} + \delta$ to x

Output: (at the end of the stream) estimate of $F_k = \sum_{i=1}^{n} |x_i|^k$ (moment of order k ≥ 3)

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Result: $\mathbf{M} = \widetilde{\mathbf{O}}\left(\frac{\mathbf{n^{1-2/k}}}{\mathbf{P^2}}\right)$ qubits of memory

(vs. $M = \widetilde{\Theta}\left(\frac{n^{1-2/k}}{P}\right)$ classical bits of memory)

[Monemizadeh, Woodruff'10]
[Andoni, Krauthgamer, Onak'11]

Conclusion

The **mean** of a random variable X can be estimated with **multiplicative error ε** using $\widetilde{\mathbf{O}}\left(\frac{\boldsymbol{\Delta}}{\epsilon} \cdot \log^{\mathbf{3}}\left(\frac{\mathbf{M}_{\boldsymbol{\Omega}}}{\mathbf{E(X)}}\right)\right)$ **quantum samples**, given $\Delta^2 \geq \frac{\mathrm{E}(X^2)}{\mathrm{E}(X)^2}$.

Lower bound: $\Omega\left(\frac{\Delta - 1}{\epsilon}\right)$ quantum samples

or $\Omega\left(\frac{\Delta^2 - 1}{\epsilon^2}\right)$ copies of the state $S_X|0\rangle = \sum_{x \in \Omega} \sqrt{p_x}\,|\psi_x\rangle\,|x\rangle$

Open questions:

- Can we improve the complexity to O(Δ/ε) exactly?
- Lower bound for the Frequency Moments estimation problem?
- Other applications ?

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