Quantum Chebyshev's Inequality and Applications

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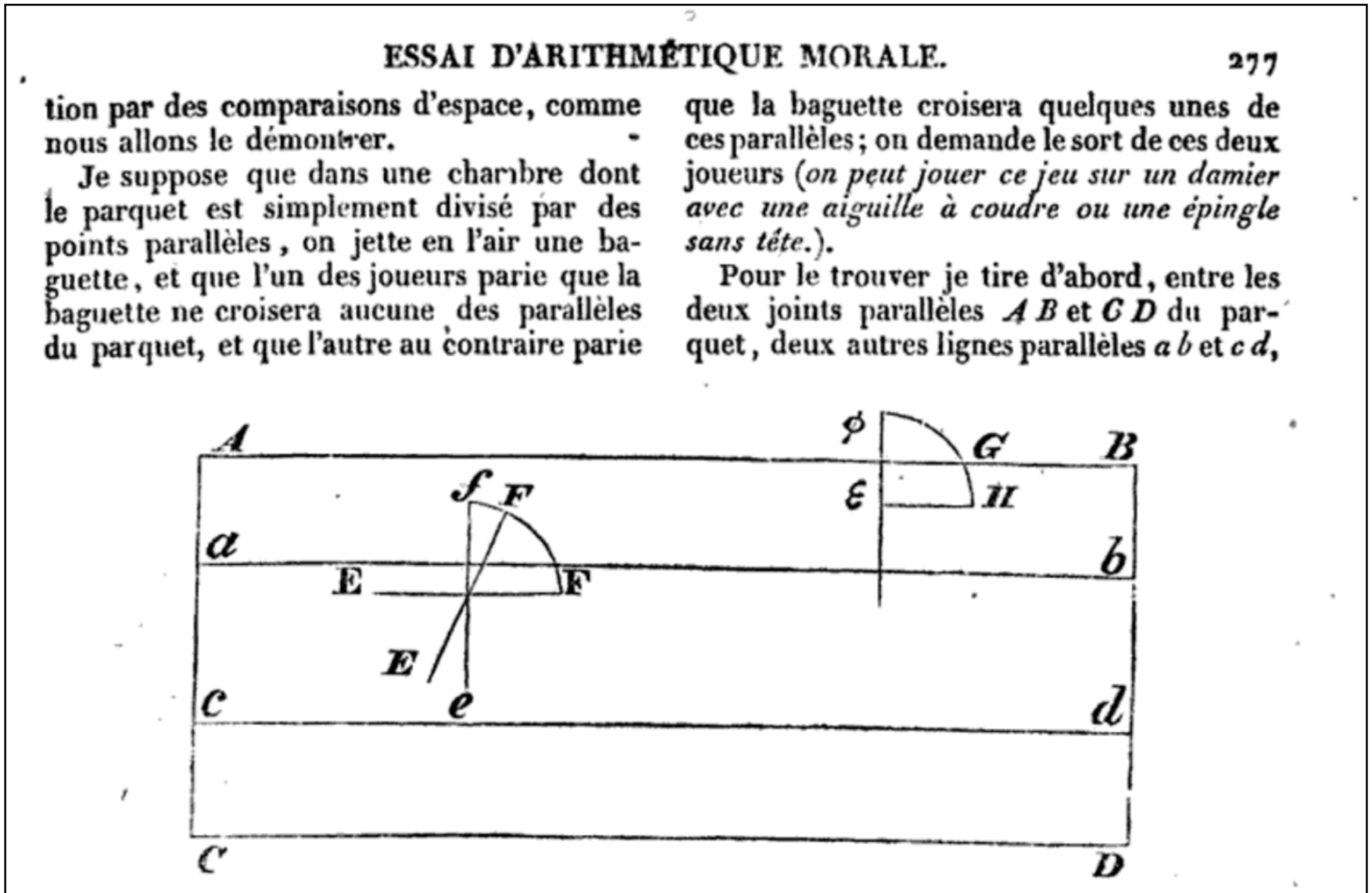
Buffon’s needle

A needle dropped randomly on a floor with equally spaced parallel lines will cross one of the lines with probability **$2/\pi$**.



ESSAI D'ARITHMÉTIQUE MORALE. 277

tion par des comparaisons d'espace, comme nous allons le démontrer.

Je suppose que dans une chambre dont le parquet est simplement divisé par des points parallèles, on jette en l'air une baguette, et que l'un des joueurs parie que la baguette ne croisera aucune des parallèles du parquet, et que l'autre au contraire parie que la baguette croisera quelques unes de ces parallèles; on demande le sort de ces deux joueurs (*on peut jouer ce jeu sur un damier avec une aiguille à coudre ou une épingle sans tête.*).

Pour le trouver je tire d'abord, entre les deux joints parallèles *A B* et *C D* du parquet, deux autres lignes parallèles *a b* et *c d*,

Buffon, G., *Essai d'arithmétique morale*, 1777.

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Law of large numbers: $\dfrac{x_1 + \ldots + x_n}{n} \xrightarrow{n \to \infty} \mathbf{E}(X)$

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multiplicative error 0 < ε < 1

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In practice: given an upper-bound $\Delta^2 \geq \dfrac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2}$, take $n = \Omega\left(\dfrac{\Delta^2}{\epsilon^2}\right)$ samples

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   Output **n*deg(v)**

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Consequence: $O(\sqrt{n}/\varepsilon^2)$ samples to approximate m with error ε.

Counting with Markov chain Monte Carlo methods:

Counting vs. sampling [Jerrum, Sinclair'96] [Štefankovič et al.'09], Volume of convex bodies [Dyer, Frieze'91], Permanent [Jerrum, Sinclair, Vigoda'04]

Data stream model:

Frequency moments, Collision probability [Alon, Matias, Szegedy'99] [Monemizadeh, Woodruff'] [Andoni et al.'11] [Crouch et al.'16]

Testing properties of distributions:

Closeness [Goldreich, Ron'11] [Batu et al.'13] [Chan et al.'14], Conditional independence [Canonne et al.'18]

Estimating graph parameters:

Number of connected components, Minimum spanning tree weight [Chazelle, Rubinfeld, Trevisan'05], Average distance [Goldreich, Ron'08], Number of triangles [Eden et al. 17]

etc.

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Question: **can we estimate E(X) with less samples in the quantum setting?**

Previous Works

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Observation: The Grover's operator $\mathbf{G} = \mathbf{S_Y^{-1}}(I - 2|0\rangle\langle 0|)\mathbf{S_Y}(I - 2I \otimes |1\rangle\langle 1|)$

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Algorithm: 1/ Apply Phase Estimation on G for $t \geq \Omega(\sqrt{M_\Omega}/(\epsilon\sqrt{\mathbf{E}(X)}))$ steps

to get an estimate $\widetilde{\theta}$ s.t. $|\widetilde{\theta} - |\theta|| \leq 1/t$.

2/ Output $\widetilde{\mu} = M_\Omega \cdot \sin^2(\widetilde{\theta})$ as an estimate to E(X).

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…not efficient if $\mathbf{M_\Omega}$ is large (worst than the classical algorithm sometimes)

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[Montanaro'15]	$\frac{\Delta^{\mathbf{2}}}{\epsilon}$	$\Delta^2 \geq \frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2}$	
Our result	$\frac{\Delta}{\epsilon} \cdot \log^{\mathbf{3}}\left(\frac{M_\Omega}{\mathbf{E}(X)}\right)$	$\Delta^2 \geq \frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2}$	Sample space **$\Omega \subset [0, M_\Omega]$**

Our Approach

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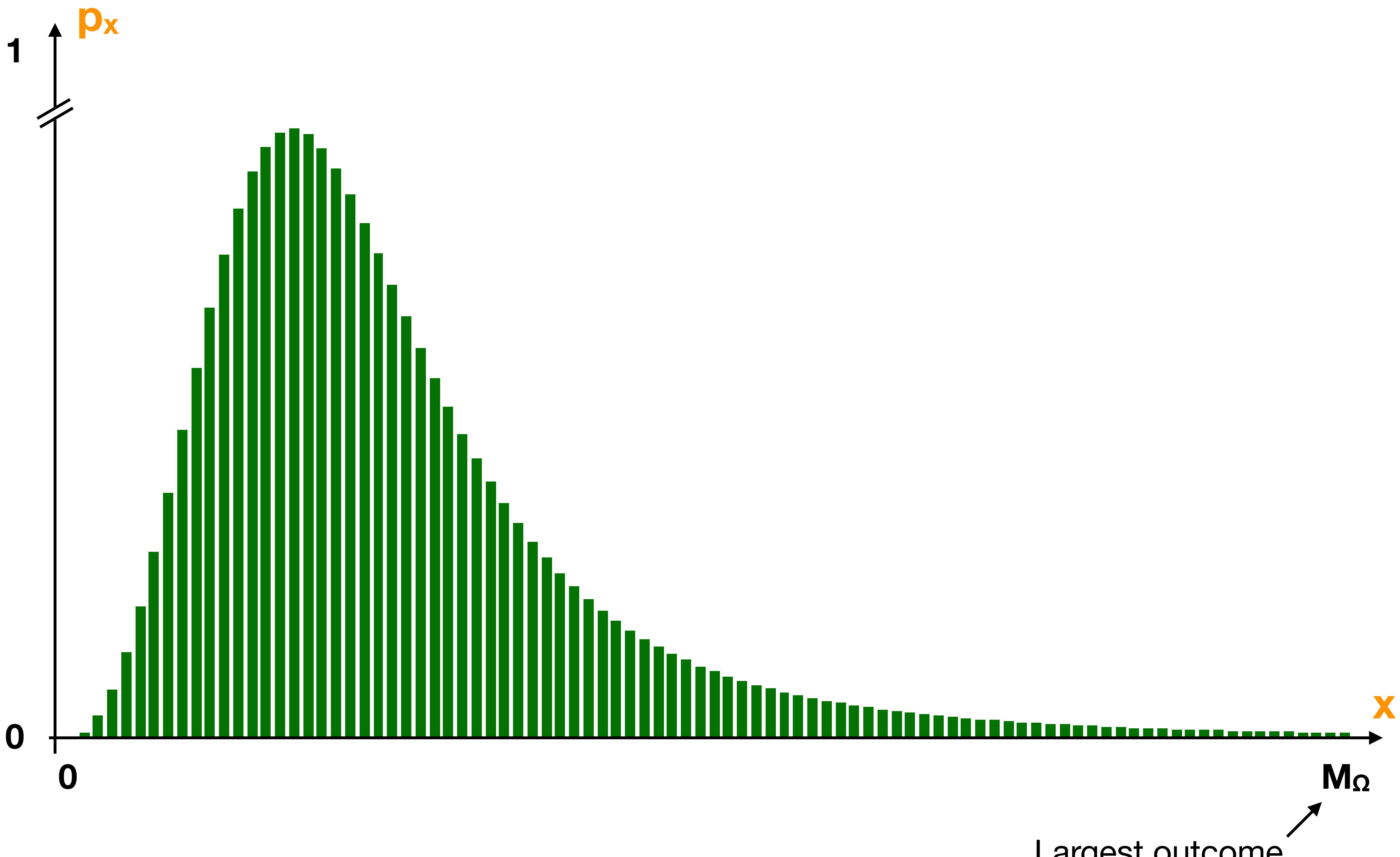
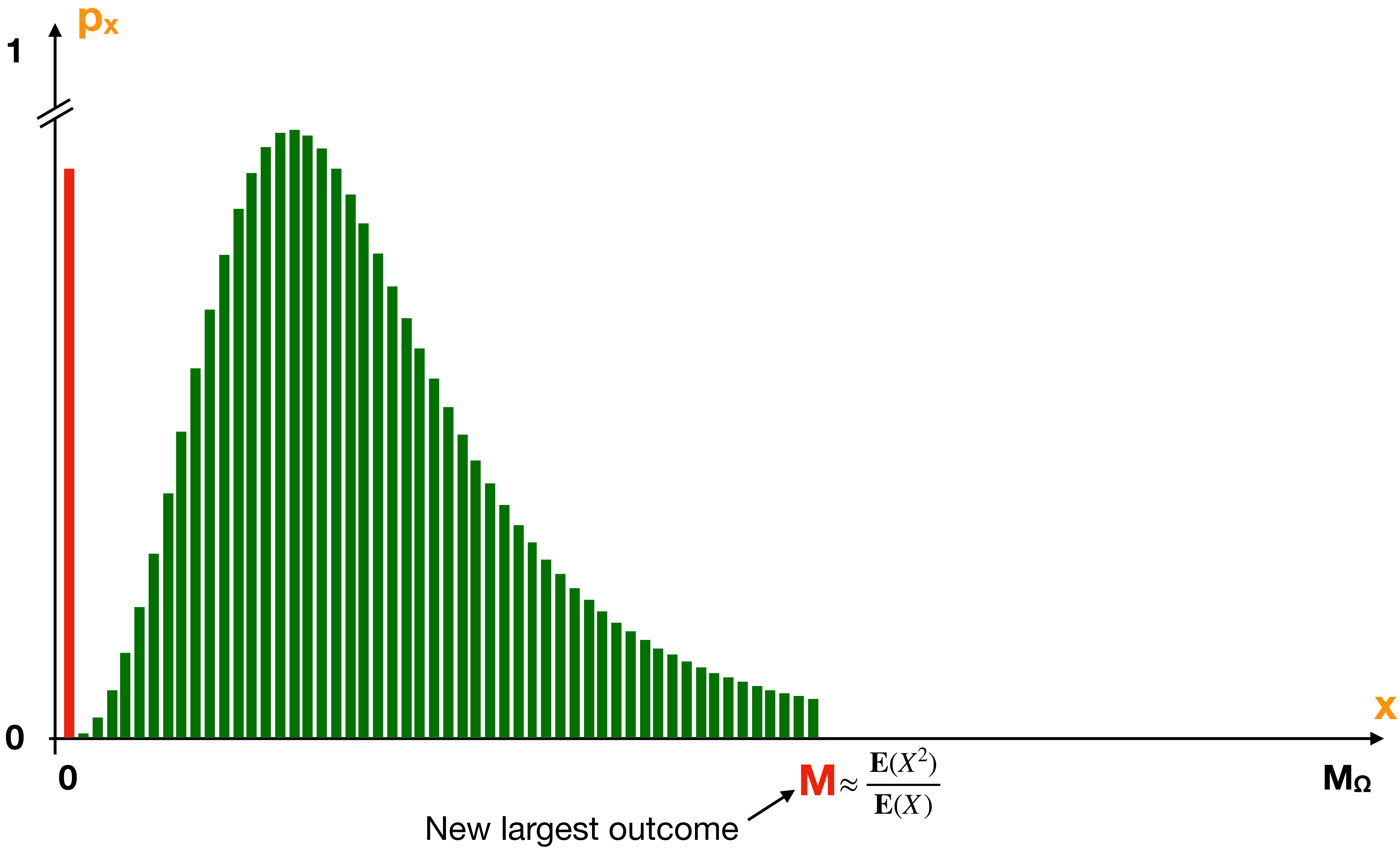


Random variable X_M



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Stopping rule: $\widetilde{\mu}_i \neq 0$ **Output:** M_i

Theorem: the first non-zero $\widetilde{\mu}_i$ is obtained w.h.p. when:

$$2 \cdot \mathbf{E}(X)\Delta^2 \leq M_i \leq 10 \cdot \mathbf{E}(X)\Delta^2$$

Analysis

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[Brassard et al.'02]

$\sqrt{\frac{\mathbf{E}(X_M)}{M}}$

$1/\Delta$

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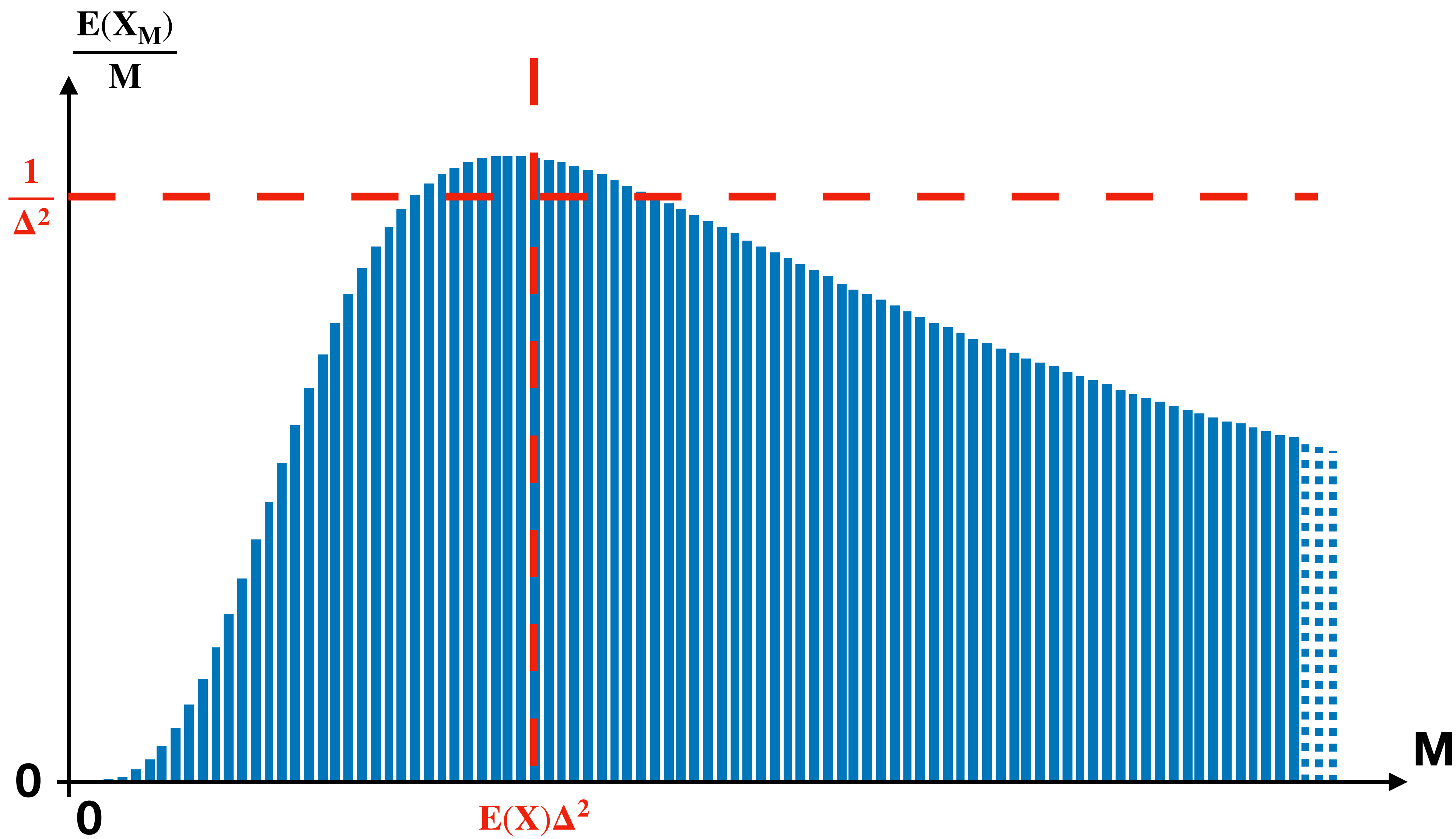

Final algorithm:

Step 1: Logarithmic search on M until **Amplitude-Estimation**$(X_M, \Delta) \neq 0$

$\longrightarrow$ $2 \cdot \mathbf{E}(X)\Delta^2 \leq M \leq 10^4 \cdot \mathbf{E}(X)\Delta^2$ with high probability

$$\Delta \cdot \log^3\left(\frac{M_\Omega}{\mathbf{E}(X)}\right)$$

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$$\Delta/\epsilon^{3/2}$$

Step 2bis: Slightly refined algorithm, adapted from [Heinrich'01, Montanaro'15]

$$\Delta/\epsilon$$

Optimality

Lower bounds

For any **Δ, ε** there exists two samplers $\begin{cases} S_X|0\rangle = \sqrt{1-p}\rangle|0\rangle + \sqrt{p}|1\rangle \\ S_Y|0\rangle = \sqrt{1-q}\rangle|0\rangle + \sqrt{q}|1\rangle \end{cases}$

with $\mathbf{E}(Y) \geq (1+2\epsilon)\cdot\mathbf{E}(X)$ and $\frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2}, \frac{\mathbf{E}(Y^2)}{\mathbf{E}(Y)^2} \in [\Delta^2, 2\Delta^2]$

such that distinguishing between X and Y requires:

$$\boldsymbol{\Omega}\left(\frac{\boldsymbol{\Delta - 1}}{\epsilon}\right)$$

Quantum samples
from S_X / S_Y

For any **Δ, ε** there exists two samplers $\begin{cases} S_X|0\rangle = \sqrt{1-p}\rangle|0\rangle + \sqrt{p}|1\rangle \\ S_Y|0\rangle = \sqrt{1-q}\rangle|0\rangle + \sqrt{q}|1\rangle \end{cases}$

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such that distinguishing between X and Y requires:

$$\Omega\left(\frac{\Delta-1}{\epsilon}\right)$$

Quantum samples from S_X / S_Y

or

$$\Omega\left(\frac{\Delta^2-1}{\epsilon^2}\right)$$

Copies of the states $S_X|0\rangle$ / $S_Y|0\rangle$

Applications

A generic quantization method

Randomized algorithm A
with output X = A()

A generic quantization method

Randomized algorithm A
with output X = A()

↓

Deterministic algorithm B with random
seed r as input and output X = B(r)

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Quantum sampler $S_X|0\rangle = \frac{1}{\sqrt{R}} \sum_{r \in [R]} |r\rangle |C(r)\rangle$

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samples to approximate E(X)

$$\frac{\Delta^2}{\epsilon^2}$$

$$\frac{\Delta}{\epsilon} \cdot \log^3\left(\frac{M_\Omega}{\mathbf{E}(X)}\right)$$

First obstacle: time complexity

Randomized algorithm A
with output X = A()

T_{max} = maximum running time of A

T_{avg} = average running time of A

Deterministic algorithm B with random seed r as input and output X = B(r)

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$\mathbf{T_{max}}$ **=** maximum running time of A

$\mathbf{T_{avg}}$ **=** average running time of A

N samples in maximum time $\mathbf{N*T_{max}}$

N samples in average time $\mathbf{N*T_{avg}}$

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T_{max} = maximum running time of A

T_{avg} = average running time of A

N samples in maximum time **$N*T_{max}$**

N samples in average time **$N*T_{avg}$**

N quantum samples in maximum/average time O(**$N*T_{max}$**)

First obstacle: time complexity

New tool: Variable-Time Amplitude Estimation

(≠ Variable-Time Amplitude Amplification)

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Randomized algorithm A
with output X in time T_{max},T_{avg}

↓

Estimate of E(X) in (average) time:

$$\frac{\Delta^2}{\epsilon^2} \cdot T_{avg}$$

First obstacle: time complexity

New tool: Variable-Time Amplitude Estimation

(≠ Variable-Time Amplitude Amplification)

Randomized algorithm A
with output X in time T_{max}, T_{avg}

Quantum sampler S_X

Estimate of E(X) in (average) time:

$$\frac{\Delta^2}{\epsilon^2} \cdot T_{avg}$$

Estimate of E(X) in time:

$$\frac{\Delta}{\epsilon^2} \cdot T_{avg,2} \cdot \mathrm{polylog}\left(\frac{M_\Omega}{\mathbf{E}(X)}, T_{max}\right)$$

where $T_{avg,2}$ = L_2-average running time of A

Application: triangle counting

Input: graph G=(V,E) with **n** vertices, **m** edges, **t** triangles

Query access: unitaries

$$O_{\mathrm{deg}}|v\rangle|0\rangle = |v\rangle|\deg(v)\rangle$$ *(degree query)*

$$O_{\mathrm{pair}}|v\rangle|w\rangle|0\rangle = |v\rangle|w\rangle|(v,w) \in E\ ?\rangle$$ *(pair query)*

$$O_{\mathrm{ngh}}|v\rangle|i\rangle|0\rangle = |v\rangle|i\rangle|v_i\rangle$$ *(neighbor query)*

i^{th} neighbor of v

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Result: $\widetilde{\Theta}\left(\frac{\sqrt{n}}{t^{1/6}} + \frac{m^{3/4}}{\sqrt{t}}\right)$ degree/pair/neighbor quantum queries to approximate **t**

(vs. $\widetilde{\Theta}\left(\frac{n}{t^{1/3}} + \frac{m^{3/2}}{t}\right)$ classical degree/pair/neighbor queries)

[Eden, Levi, Ron'15] [Eden, Levi, Ron, Seshadhri'17]

Second obstacle: reversibility and streaming algorithms

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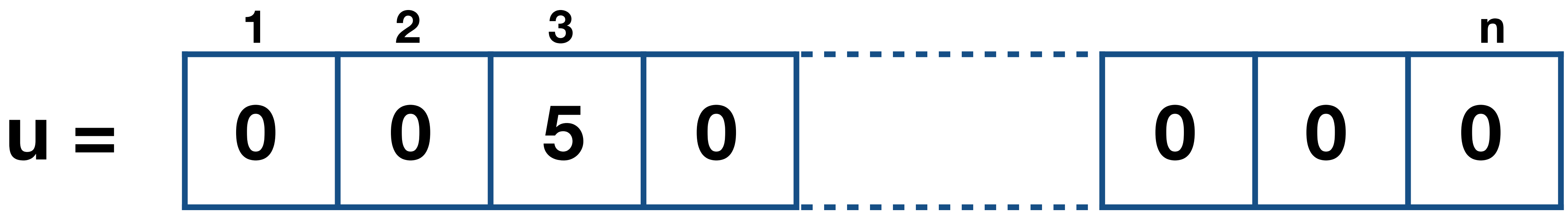
Second obstacle: reversibility and streaming algorithms

	1	2	3					n
u =	**0**	**0**	**0**	**0**	…	**0**	**0**	**0**

Stream of **updates** to u:

Second obstacle: reversibility and streaming algorithms



Stream of **updates** to u: **(3,+5)**

Second obstacle: reversibility and streaming algorithms

u =

1	2	3					n
0	**-6**	**5**	**0**	…	**0**	**0**	**0**

Stream of **updates** to u: **(3,+5) ; (2,-6)**

Second obstacle: reversibility and streaming algorithms

u =

1	2	3					n
0	-6	4	0	...	0	0	0

Stream of **updates** to u: **(3,+5) ; (2,-6) ; (3,-1)**

Second obstacle: reversibility and streaming algorithms

	1	2	3		…			n
u =	**0**	**-6**	**4**	**0**	…	**0**	**0**	**0**

Stream of **updates** to u: **(3,+5) ; (2,-6) ; (3,-1)**

Goal: approximate some function f(u) of the **final** vector u

(example: f(u) = # of distinct elements in u)

Second obstacle: reversibility and streaming algorithms

$$u = \begin{array}{cccccccc} 1 & 2 & 3 & & & & & n \\ \boxed{0} & \boxed{-6} & \boxed{4} & \boxed{0} & \cdots & \boxed{0} & \boxed{0} & \boxed{0} \end{array}$$

Stream of **updates** to u: **(3,+5) ; (2,-6) ; (3,-1)**

Goal: approximate some function f(u) of the **final** vector u

(example: f(u) = # of distinct elements in u)

Algorithm with smallest possible **memory M ≪ n** using **P passes** over the same stream to approximate f(u)?

Second obstacle: reversibility and streaming algorithms

	1	2	3		…			n
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Stream of **updates** to u: **(3,+5) ; (2,-6) ; (3,-1)**

Goal: approximate some function f(u) of the **final** vector u

(example: f(u) = # of distinct elements in u)

Algorithm with smallest possible **memory M** $\ll$ **n** using **P passes** over the same stream to approximate f(u)?

Standard method (Alon, Matias, Szegedy'99):

Design an algorithm A with memory **M** that produces in **1 pass** a sample X = A(1 pass) such that **E(X) = f(u)** and $E(X^2)/E(X)^2 \leq P$

$\longrightarrow$ the average of P samples over P passes is a good approximation of f(u)

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- Our algorithm needs S_X^{-1}, which requires to run C^{-1}.

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We showed that **linear sketch** streaming algorithms can be made reversible efficiently.

(= C^{-1} and S_X^{-1} can be implemented with one pass in the **direct** direction)

Application: frequency moments in the streaming model

u =

1	2	3		…			n
0	**-6**	**5**	**0**	…	**0**	**0**	**0**

Frequency moment of order k ≥ 3: $f_k(u) = \sum_{i=1}^{n} |u_i|^k$

Application: frequency moments in the streaming model

	1	2	3					n
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Best **P-pass** algorithm with **memory M** approximating f_k?

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Best **P-pass** algorithm with **memory M** approximating f_k?

Classically: PM = $\Theta(n^{1-2/k})$

1 pass + memory $\mathbf{M = \frac{n^{1-2/k}}{P}}$

||

1 sample from a random variable X with

$\mathbf{E(X) \approx f_k(u)}$ and $\mathbf{E(X^2)/E(X)^2 \leq P}$

[Monemizadeh, Woodruff'10]
[Andoni, Krauthgamer, Onak'11]

Application: frequency moments in the streaming model

	1	2	3					n
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[Monemizadeh, Woodruff'10]
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Quantumly: $P^2M = O(n^{1-2/k})$

1 pass + memory $\mathbf{M = \frac{n^{1-2/k}}{P^2}}$

||

1 **quantum** sample S_X from a r.v. X with

$\mathbf{E(X) \approx f_k(u)}$ and $\mathbf{E(X^2)/E(X)^2 \leq P^2}$

Conclusion

The **mean** of a random variable X can be estimated with **multiplicative error ε** using $\widetilde{\mathbf{O}}\left(\frac{\boldsymbol{\Delta}}{\epsilon} \cdot \log^{\mathbf{3}}\left(\frac{\mathbf{M}_{\boldsymbol{\Omega}}}{\mathbf{E}(\mathbf{X})}\right)\right)$ **quantum samples**, given $\Delta^2 \geq \frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2}$.

Open questions:

- Can we improve the complexity to O(Δ/ε) ?
- Sample space Ω with negative values?
- Lower bound for the Frequency Moments estimation problem?

  (would follow from an $\Omega(t + nm^{-2}/t)$ lower bound for the 2-player **t**-round cc of L_∞ problem)

- Other applications ?

arXiv: 1807.06456

Extra slides

No a priori information on $E(X^2)/E(X)^2$

Result: There is an **optimal** algorithm that approximates the mean of any quantum sampler S_X over $\Omega \subset [0,B]$ with

$$\widetilde{\Theta}\left(\frac{\sqrt{B}}{\sqrt{\epsilon \mathbf{E}(X)}} + \frac{\mathbf{E}(X^2)}{\epsilon \mathbf{E}(X)}\right)$$

quantum samples, when there is no a priori information on X.

→ Quantization of [Dagum, Karp, Luby, Ross'00]

Lemma: If $b \geq \dfrac{\mathbf{E}(X^2)}{\epsilon \mathbf{E}(X)}$ then $(1-\epsilon)\mathbf{E}(X) \leq \mathbf{E}(X_{<b}) \leq \mathbf{E}(X)$.

Lemma: If $b \geq 10^4 \cdot \mathbf{E}(X)\Delta^2$ then $\dfrac{\mathbf{E}(X_{<b})}{b} \leq \dfrac{1}{10^4 \cdot \Delta^2}$

Lemma: If $b \geq \frac{\mathbf{E}(X^2)}{\epsilon \mathbf{E}(X)}$ then $(1-\epsilon)\mathbf{E}(X) \leq \mathbf{E}(X_{<b}) \leq \mathbf{E}(X)$.

Proof:

- $\mathbf{E}(X_{\geq b}) \leq \frac{\mathbf{E}(X^2)}{b} \leq \epsilon \mathbf{E}(X)$
- $\mathbf{E}(X_{<b}) = \mathbf{E}(X) - \mathbf{E}(X_{\geq b}) \geq (1-\epsilon)\mathbf{E}(X)$

Lemma: If $b \geq 10^4 \cdot \mathbf{E}(X)\Delta^2$ then $\frac{\mathbf{E}(X_{<b})}{b} \leq \frac{1}{10^4 \cdot \Delta^2}$

Proof: $\frac{\mathbf{E}(X_{<b})}{b} \leq \frac{\mathbf{E}(X)}{10^4 \mathbf{E}(X)\Delta^2} \leq \frac{1}{10^4 \cdot \Delta^2}$