Near-optimal Quantum Algorithms for Multivariate Mean Estimation

Problem setup

1. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
2. A *d*-dimensional random variable $X : \Omega \to \mathbb{R}^d$.

Properties:

- *Mean:* $\mu = \mathbb{E}[X] = \sum_{\omega \in \Omega} \mathbb{P}(\omega) X(\omega) \in \mathbb{R}^d$.
- *Covariance matrix:* $\Sigma = \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1, X_2] & \cdots & \mathrm{Cov}[X_1, X_d] \\ \mathrm{Cov}[X_1, X_2] & \mathrm{Var}[X_2] & \cdots & \mathrm{Cov}[X_2, X_d] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}[X_1, X_d] & \mathrm{Cov}[X_2, X_d] & \cdots & \mathrm{Var}[X_d] \end{bmatrix}$.
- *Finite trace:* $\mathrm{Tr}[\Sigma] = \sum_{j=1}^{d} \mathrm{Var}[X_j] < \infty$.

Access models

Classical access model:

1. Obtain outcome $\omega \sim \mathbb{P}$.
2. Evaluate function $\omega \mapsto X(\omega)$.

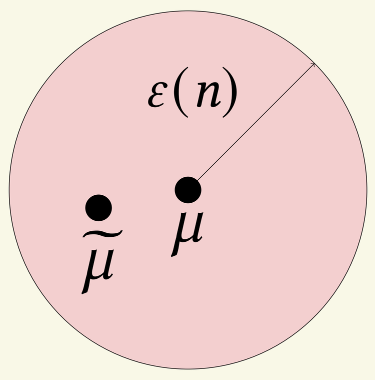
Quantum access model:

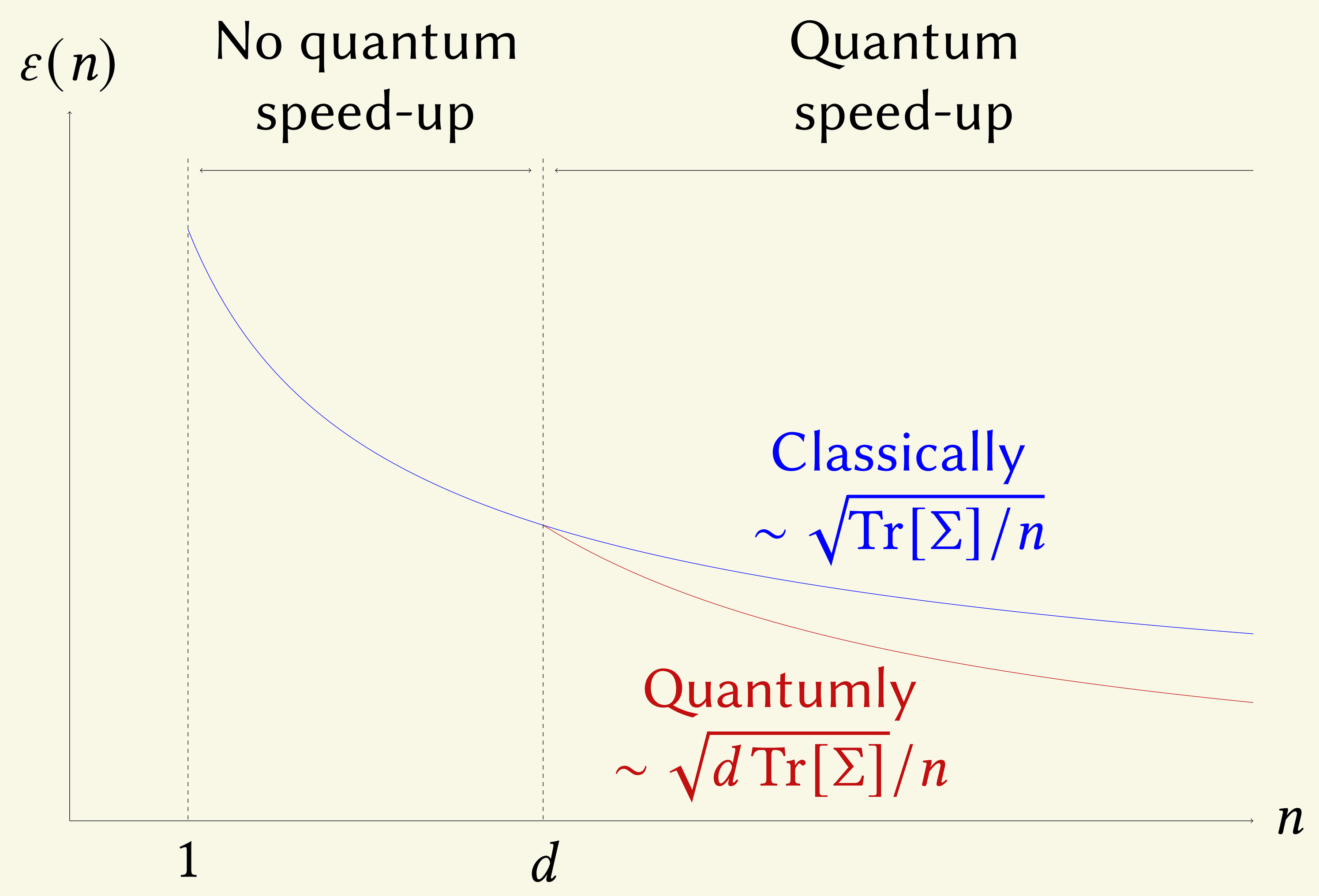
1. Distribution oracle: $|0\rangle \mapsto \sum_{\omega \in \Omega} \sqrt{\mathbb{P}(\omega)} \, |\omega\rangle$.
2. Evaluation oracle: $|\omega\rangle \, |0\rangle \mapsto |\omega\rangle \, |X(\omega)_1\rangle \otimes \cdots \otimes |X(\omega)_d\rangle$.

↳ Calls to these routines are *samples*.

Results

Goal: Construct an estimator $\widetilde{\mu}$, using n samples, s.t. $\mathbb{P}\left[\|\mu - \widetilde{\mu}\|_2 \leq \varepsilon(n)\right] \geq \frac{2}{3}$.

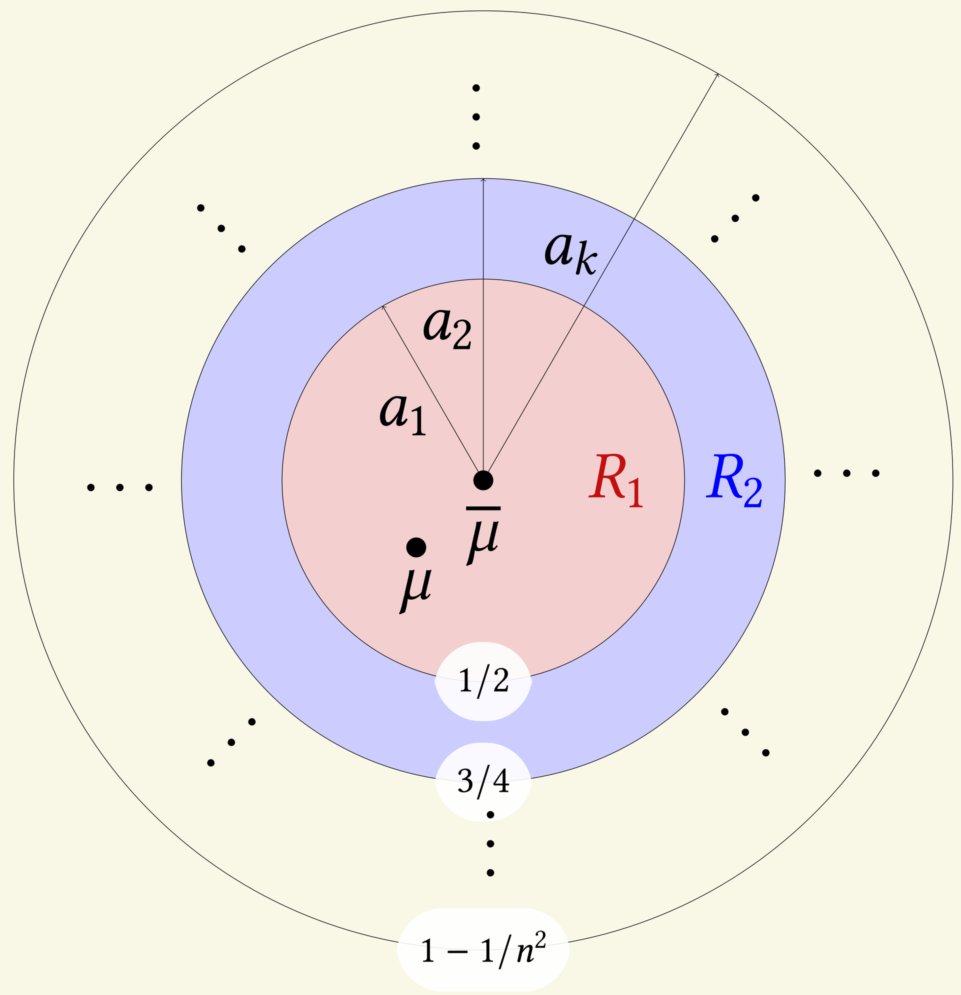




Crucial observation: quantum speed-up only when $n \geq d$.

Quantum estimator outline

1. *Get a crude estimate:* $\overline{\mu}$ s.t. $\|\mu - \overline{\mu}\|_2 \leq \sqrt{\mathrm{Tr}[\Sigma]}$.
2. *Get an idea of the spread:* Estimate quantiles a_ℓ s.t. $\mathbb{P}\left[\|X - \overline{\mu}\|_2 \geq a_\ell\right] \approx \frac{1}{2^\ell}$, for $\ell \in \{1, \ldots, 2\log(n)\}$.
3. *Estimate truncated mean on the rings:* $\mathbb{E}[X] \approx \sum_{\ell=1}^{k} \mathbb{E}[X^{(\ell)}]$.



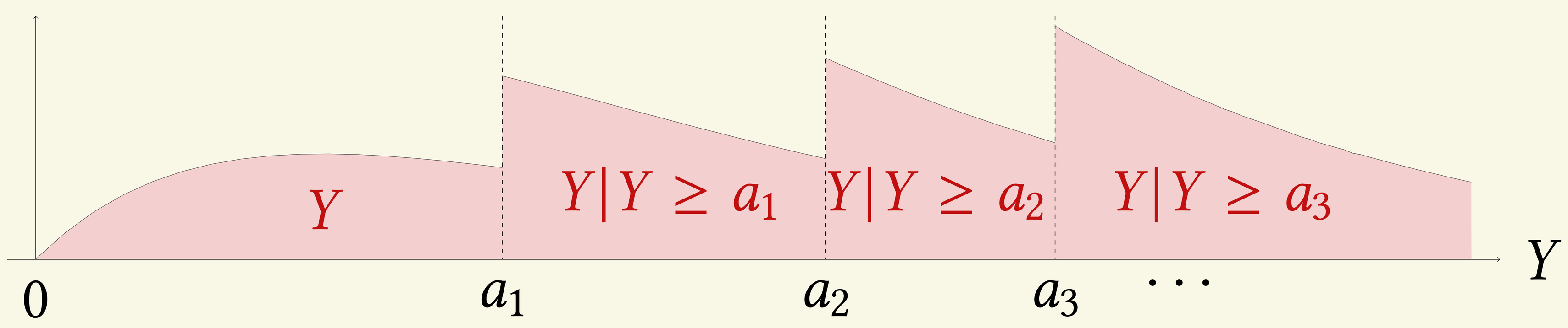
Quantile estimation

Define the univariate random variable $Y = \|X - \overline{\mu}\|_2$.

Observation: $a_\ell \approx$ median of $\widetilde{O}(1)$ samples from $Y | Y \geq a_{\ell-1}$.

↓

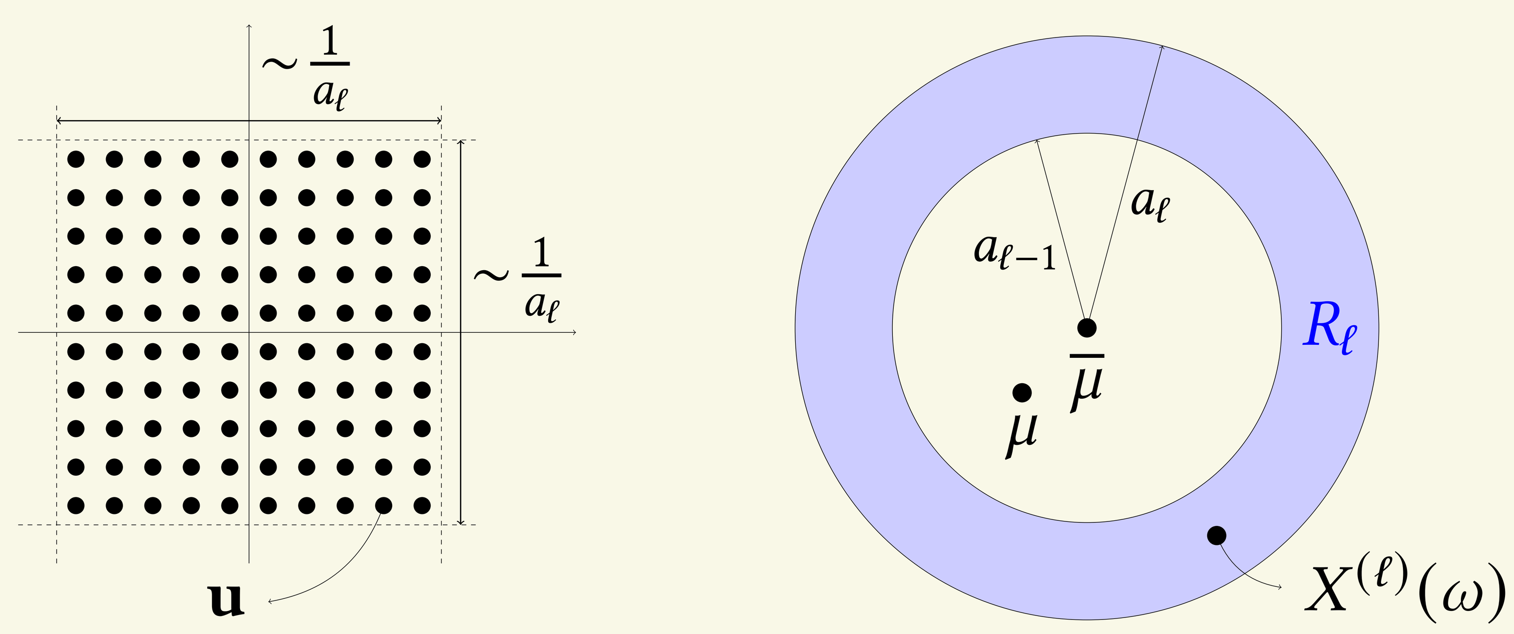
Rejection sampling via *amplitude amplification* requires $\widetilde{O}\big(1/\sqrt{\mathbb{P}[Y \geq a_{\ell-1}]}\big) = \widetilde{O}\big(\sqrt{2^\ell}\big)$ samples from Y.



Truncated mean estimation on the rings

1. *Amplitude amplification on the R_ℓ ring:* $\widetilde{O}\big(\sqrt{2^\ell}\big)$ samples.
2. *Phase encoding techniques:* $\widetilde{O}\big(n/\sqrt{2^\ell}\big)$ calls to 1.

   Directional mean oracle: $|\mathbf{u}\rangle \mapsto e^{in2^{\ell/2} \cdot \mathbf{u}^T \mathbb{E}[X^{(\ell)}]} \, |\mathbf{u}\rangle$.



3. *Bernstein-Vazirani over the reals:* $\widetilde{O}(1)$ calls to 2.

   Apply QFT^{-1} on $\sum_u e^{in2^{\ell/2} \cdot \mathbf{u}^T \mathbb{E}[X^{(\ell)}]} \, |\mathbf{u}\rangle$ and measure.

$$\left\|\widetilde{\mu}^{(\ell)} - \mathbb{E}[X^{(\ell)}]\right\| = \widetilde{O}\left(\sqrt{d \cdot a_\ell \cdot \mathbb{E}\|X^{(\ell)}\|/n}\right) = \widetilde{O}\left(\sqrt{d \cdot \mathrm{Tr}[\Sigma]/n}\right)$$

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