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Quantum Query Complexity Problem Session 4

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Problem 1

Question 1

The statement that $\mathrm{Adv}^*(f) \leq \mathrm{Adv}(f)$ will follow from the following two observations.

1. The feasible region of the second program is a superset of the feasible region of the first program. Since both programs are minimizing, having a larger feasible region means that a smaller quantity may appear and decrease the optimum of the second program.

   We can see that the feasible region is a superset, since the first program has the constraint that for all x, y

   $$\sum_{i:\ x_i \neq y_i} \left\langle v^{(x,i)} \middle| v^{(y,i)} \right\rangle = \begin{cases} 1 & f(x) \neq f(y) \\ 0 & f(x) = f(y) \end{cases} .$$

   The second program only constraints the sum of the inner products to be equal to 1 if $f(x) \neq f(y)$, but otherwise the sum of the inner products can be any arbitrary value.

2. The objective function of the second program is at most as large as the objective function of the first program at every point.

   This is because if we use the variables of the second program, the first objective equals $\max\{C_0, C_1\}$. Then it holds that

   $$\max\{C_0, C_1\} = \sqrt{\max\{C_0, C_1\}^2} \geq \sqrt{C_0 C_1}.$$

Question 2

Given a feasible solution $\{v^{(x,i)}\}$ to the first program, we will construct another feasible solution with value $\sqrt{C_0C_1}$. This solution will satisfy

$$u^{(x,i)} = \left(\frac{C_1}{C_0}\right)^{1/4} \cdot v^{(x,i)} \text{ for } f(x) = 0,$$

$$u^{(x,i)} = \left(\frac{C_0}{C_1}\right)^{1/4} \cdot v^{(x,i)} \text{ for } f(x) = 1.$$

We can verify that

$$\begin{aligned}\sum_{i:x_i\neq y_i} \left\langle u^{(x,i)} | u^{(y,i)} \right\rangle &= \sum_{i:x_i\neq y_i} \left(\frac{C_0}{C_1}\right)^{1/4} \cdot \left(\frac{C_1}{C_0}\right)^{1/4} \cdot \left\langle v^{(x,i)} | v^{(y,i)} \right\rangle \\ &= \sum_{i:x_i\neq y_i} \cdot \left\langle v^{(x,i)} | v^{(y,i)} \right\rangle \\ &= \mathbf{1}_{f(x)\neq f(y)}.\end{aligned}$$

Additionally,

$$\max_{x:f(x)=0} \sum_i \|u^{(x,i)}\|^2 = \sqrt{\frac{C_1}{C_0}} \cdot \max_{x:f(x)=0} \sum_i \|v^{(x,i)}\|^2 = \sqrt{C_0C_1},$$

$$\max_{x:f(x)=1} \sum_i \|u^{(x,i)}\|^2 = \sqrt{\frac{C_0}{C_1}} \cdot \max_{x:f(x)=1} \sum_i \|v^{(x,i)}\|^2 = \sqrt{C_0C_1}.$$

Thus the value of the program for $u^{(x,i)}$ is $\sqrt{C_0C_1}$.

Question 3

$$\sum_{i:x_i\neq y_i} \left\langle w^{(x,i)} | w^{(y,i)} \right\rangle = \sum_{i:x_i\neq y_i} \left\langle v^{(x,i)}, x_i \oplus f(x) | v^{(y,i)}, y_i \oplus f(y) \right\rangle$$

If $f(x) = f(y)$, then $x_i \oplus f(x) \neq y_i \oplus f(y)$, and thus the inner product is zero for all i. If on the other hand, $f(x) \neq f(y)$, then $x_i \oplus f(x) = y_i \oplus f(y)$ and thus the summation becomes

$$= \sum_{i:x_i\neq y_i} \left\langle v^{(x,i)} | v^{(y,i)} \right\rangle,$$

which is equal to 1 since $\{v^{(x,i)}\}$ is a feasible solution to the second program.

Question 4

Let $v^{(x,i)}$ be such that achieves the minimum value of $\sqrt{C_0 C_1}$ in the second program. Then we define $w^{(x,i)}$ as in Question 3, which implies that $w^{(x,i)}$ is a feasible solution to the first program. Additionally, note that $\|w^{(x,i)}\| = \|v^{(x,i)}\|$. Thus the values of C_0 and C_1 in the first program for the w variables are the same as the C_0, C_1 in the second program for the v variables. Now we use Question 2 and obtain that there exists a feasible solution to the first program that obtains the value $\sqrt{C_0 C_1}$.

Thus the minimum value of the first program is at least as small as the minimum value of the second program $\implies \mathrm{Adv}^*(f) \geq \mathrm{Adv}(f)$. Finally, from Question 1 we conclude the equality.

Problem 2

Question 1

We only need to consider the edges $\{i,j\}$ that are on the st-path used to construct $\left|w^{(y,\{i,j\})}\right\rangle$, since for the other edges $\langle w^{(x,\{i,j\})}|w^{(y,\{i,j\})}\rangle = 0$.

Fix any st-path $i_1 = s \to i_2 \to \ldots \to t$ of shortest length in y. The quantity $\langle w^{(x,\{i_\ell,i_{\ell+1}\})}|w^{(y,\{i_\ell,i_{\ell+1}\})}\rangle$ is $+1$ when $i_\ell \in V_x(1)$, $i_{\ell+1} \notin V_x(1)$ and -1 when $i_\ell \notin V_x(1)$, $i_{\ell+1} \in V_x(1)$. It is 0 otherwise. Since the path starts in $1 \in V_x(1)$ ends at $t \notin V_x(1)$, we must have one more edge in the first case than in the second. Thus the summation is equal to 1 as desired.

Question 2

We construct a solution $\left|v^{(x,\{i,j\})}\right\rangle \in \operatorname{span}\{|k\rangle|t\rangle : 1 \le k \le n, 2 \le t \le n\}$ to the dual adversary, where the extra register $|t\rangle$ will be used to embed the st-Connectivity problem with $s = 1$ and all t as follows.

Define $\mathcal{G}'_0$ as the set of input graphs that are not connected, and $\mathcal{G}'_1$ as the graphs that are connected.

If $x \in \mathcal{G}'_0$, observe first that $\overline{V_x(1)} \neq \varnothing$ since at least one vertex doesn't belong to the connected component of 1. Define:

$$\left|v^{(x,\{i,j\})}\right\rangle = \frac{1}{|\overline{V_x(1)}|} \sum_{t \in \overline{V_x(1)}} \left|w_t^{(x,\{i,j\})}\right\rangle |t\rangle$$

where $\left|w_t^{(x,\{i,j\})}\right\rangle$ is the vector analyzed in question 1 for st-connectivity (actually $\left|w_t^{(x,\{i,j\})}\right\rangle = \left|w^{(x,\{i,j\})}\right\rangle$ since it doesn't depend on t when $x \in \mathcal{G}'_0$).

If $x \in \mathcal{G}'_1$ then we similarly take:

$$\left|v^{(x,\{i,j\})}\right\rangle = \sum_{t \in \{2,\ldots,n\}} \left|w_t^{(x,\{i,j\})}\right\rangle |t\rangle .$$

By construction, for all $x \in \mathcal{G}'_0$, $y \in \mathcal{G}'_1$, we have: $\sum_{\{i,j\}:x_{\{i,j\}} \neq y_{\{i,j\}}} \langle v^{(x,\{i,j\})} | v^{(y,\{i,j\})} \rangle = \sum_{t \in \overline{V_x(1)}} \frac{1}{|\overline{V_x(1)}|} \sum_{\{i,j\}:x_{\{i,j\}} \neq y_{\{i,j\}}} \langle w_t^{(x,\{i,j\})} | w_t^{(y,\{i,j\})} \rangle = \sum_{t \in \overline{V_x(1)}} \frac{1}{|\overline{V_x(1)}|} \cdot 1 = 1$, using the result shown in question 1.

It remains to prove that the value of that solution is $O(n^{3/2})$. One can easily check that $C_1 = O(n^2)$ since the shortest length st-paths are always of size at most $n-1$. We show that $C_0 = O(n)$. Fix any $x \in \mathcal{G}'_0$. Observe that $\sum_{\{i,j\}} \| w_t^{(x,\{i,j\})} \|^2 = O(n|\overline{V_x(1)}|)$ since the only edges that can contribute to the sum are those going from $V_x(1)$ to $\overline{V_x(1)}$. Hence, $\sum_{\{i,j\}} \| v^{(x,\{i,j\})} \|^2 = O(\frac{1}{|\overline{V_x(1)}|^2} \cdot |\overline{V_x(1)}| \cdot n |\overline{V_x(1)}|) = O(n)$. Thus $\sqrt{C_0 C_1}$ for our construction of the v variables is equal to $O(n^{3/2})$.

Problem 3

Question 1

We will use the candidate dual adversary solution provided in the hint. Then we can compute for $f \circ g(X) = 0$ and $f \circ g(Y) = 1$:

$$\begin{aligned}
& \sum_{\substack{(i,j) \\ X_{(i,j)} \neq Y_{(i,j)}}} \langle v_{f\circ g}^{(X,(i,j))} | v_{f\circ g}^{(Y,(i,j))} \rangle \\
= & \sum_i \langle v_f^{(g(X_1),\dots,g(X_n)),i)} | v_f^{(g(Y_1),\dots,g(Y_n)),i)} \rangle \sum_{\substack{j \\ X_{(i,j)} \neq Y_{(i,j)}}} \langle v_g^{(X_i,j)} | v_g^{(Y_i,j)} \rangle \\
= & \sum_i \langle v_f^{(g(X_1),\dots,g(X_n)),i)} | v_f^{(g(Y_1),\dots,g(Y_n)),i)} \rangle \cdot \mathbf{1}[g(X_i) \neq g(Y_i)] \\
= & \sum_{\substack{i \\ g(X_i) \neq g(Y_i)}} \langle v_f^{(g(X_1),\dots,g(X_n)),i)} | v_f^{(g(Y_1),\dots,g(Y_n)),i)} \rangle \\
= & \mathbf{1}[f \circ g(X) \neq f \circ g(Y)]
\end{aligned}$$

Thus the candidate solution is a feasible point. Let us now compute its value:

$$\begin{aligned}
\sum_{(i,j)} \left\| v_{f\circ g}^{(X,(i,j))} \right\|^2 &= \max_X \sum_{(i,j)} \langle v_f^{(g(X_1),\dots,g(X_n)),i)} | v_f^{(g(Y_1),\dots,g(Y_n)),i)} \rangle \cdot \langle v_g^{(X_j,j)} | v_g^{(Y_i,j)} \rangle \\
&\leq \max_X \sum_i \langle v_f^{(g(X_1),\dots,g(X_n)),i)} | v_f^{(g(Y_1),\dots,g(Y_n)),i)} \rangle \cdot \max_X \sum_j \langle v_g^{(X_i,j)} | v_g^{(Y_i,j)} \rangle \\
&\leq \max_X \sum_i \langle v_f^{(g(X_1),\dots,g(X_n)),i)} | v_f^{(g(Y_1),\dots,g(Y_n)),i)} \rangle \cdot \mathrm{Adv}(g) \\
&= \mathrm{Adv}(f) \cdot \mathrm{Adv}(g)
\end{aligned}$$

Where for the last two equalities we used the fact that v_f, v_g are the dual adversary solutions for f and g. Thus we have constructed a feasible solution for the $f \circ g$ function that has a value at most $\mathrm{Adv}(f) \cdot \mathrm{Adv}(g)$. Since the adversary program is minimizing, its optimal value may be even lower, and thus the statement holds.

Question 2

As per the hint, let Γ be the primal adversary solution for g. We will construct a primal adversary solution for $f \circ g$ as a block matrix. For simplicity, define $G_0 = \{X \mid g(X) = 0\}$ and $G_1 = \{Y \mid g(Y) = 1\}$. Then our new adversary matrix Γ' will have blocks that are indexed as n-tuples of G_0, G_1. In particular,

$$\Gamma'[(i_1, \ldots, i_n), (j_1, \ldots, j_n)] = \begin{cases} \Gamma & \text{if } \{\sum_k i_k, \sum_k j_k\} = \{0, 1\} \\ \mathbf{0} & \text{o.w.} \end{cases}$$

This is just the adversary matrix we saw for the OR function, but each block now includes the adversary matrix Γ for g.

One can verify that Γ' is a valid adversary matrix since it is symmetric and is equal to zero for inputs that map to the same value.

It remains for us to bound

$$\frac{\|\Gamma'\|}{\max_i \|\Gamma'_i\|}.$$

It is easy to see that $\|\Gamma'\| = \sqrt{n} \cdot \|\Gamma\|$ since it is a block matrix with n copies of Γ in each block.

Let us now investigate the form of Γ'_i. Here i is a coordinate from 1 to mn. Coordinate i lies in the r^{th} block and c^{th} coordinate of the block, where $r = \lceil i/m \rceil$ and $c = i \mod m + 1$.

Then Γ'_i will have all blocks equal to $\mathbf{0}$, except possibly block $(0, \ldots, 0)$ and $(0, \ldots, 1, \ldots, 0)$, where the 1 is in the r^{th} position. This block will be equal to Γ_c. Thus $\|\Gamma'_i\| = \|\Gamma_c\|$, due to the block format of the big matrix. Thus

$$\text{Adv}(f \circ g) = \frac{\|\Gamma'\|}{\max_i \|\Gamma'_i\|} \geq \frac{\sqrt{n} \cdot \|\Gamma\|}{\max_{c=1}^m \|\Gamma_c\|} = \sqrt{n} \cdot \text{Adv}(g).$$