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Quantum Query Complexity Problem Session 3

Instructor: Yassine Hamoudi Teaching Assistant: Angelos Pelecanos

Problem 1

Question 1

Consider a randomized algorithm that outputs some value i. Then the algorithm wins if

- i is in the record and $x_i = 1$. This happens with probability at most Δ_T, or
- i is not in the record, and when x_i is sampled, it records 1. This happens with probability $\frac{1}{n}$.

From the union-bound inequality, the probability that the randomized algorithm succeeds is at most the sum of the probabilities of the two events, which is at most $\Delta_T + \frac{1}{n}$.

Thus any randomized algorithm that succeeds with probability at least $\frac{2}{3}$ must satisfy that

$$\begin{aligned} \Delta_T + \frac{1}{n} &\geq \frac{2}{3} \\ \implies \frac{T+1}{n} &\geq \frac{2}{3} \\ \implies T &= \Omega(n). \end{aligned}$$

Question 2.1

Case $x_i = \emptyset$.

$$\begin{aligned}
&\left\|\Pi_{\text{succeed}}\left(S^{\otimes n}\left|x_1,\ldots,x_i=\emptyset,\ldots,x_n\right\rangle\otimes\left|i,b\right\rangle\right)\right\| \\
&=\left\|\Pi_{\text{succeed}}\left(\frac{1}{\sqrt{n}}\sum_{0\le y<n}S\left|x_1\right\rangle,\ldots,\left|y\right\rangle,\ldots,S\left|x_n\right\rangle\otimes\left|i,b\right\rangle\right)\right\| \\
&=\left\|\frac{1}{\sqrt{n}}S\left|x_1\right\rangle,\ldots,\left|1\right\rangle,\ldots,S\left|x_n\right\rangle\otimes\left|i,b\right\rangle\right\| \\
&=\frac{1}{\sqrt{n}}.
\end{aligned}$$

Case $x_i = 1$. We will use the fact that $S\left|y\right\rangle = \left|y\right\rangle + \left|\text{err}\right\rangle$

$$\begin{aligned}
&\left\|\Pi_{\text{succeed}}\left(S^{\otimes n}\left|x_1,\ldots,x_i=1,\ldots,x_n\right\rangle\otimes\left|i,b\right\rangle\right)\right\| \\
&=\left\|\Pi_{\text{succeed}}\left(S\left|x_1\right\rangle,\ldots,(\left|1\right\rangle+\left|\text{err}\right\rangle),\ldots,S\left|x_n\right\rangle\otimes\left|i,b\right\rangle\right)\right\| \\
&=\left\|\frac{n-1}{n}\left(S\left|x_1\right\rangle,\ldots,\left|1\right\rangle,\ldots,S\left|x_n\right\rangle\otimes\left|i,b\right\rangle\right)\right\| \\
&=\frac{n-1}{n}
\end{aligned}$$

Case $x_i \in \{0,\ldots,n-1\}\setminus\{1\}$. Say $x_i = y \neq 1$.

$$\begin{aligned}
&\left\|\Pi_{\text{succeed}}\left(S^{\otimes n}\left|x_1,\ldots,x_i=y,\ldots,x_n\right\rangle\otimes\left|i,b\right\rangle\right)\right\| \\
&=\left\|\Pi_{\text{succeed}}\left(S\left|x_1\right\rangle,\ldots,(\left|y\right\rangle+\left|\text{err}\right\rangle),\ldots,S\left|x_n\right\rangle\otimes\left|i,b\right\rangle\right)\right\| \\
&=\left\|-\frac{1}{n}\left(S\left|x_1\right\rangle,\ldots,\left|1\right\rangle,\ldots,S\left|x_n\right\rangle\otimes\left|i,b\right\rangle\right)\right\| \\
&=\frac{1}{n}
\end{aligned}$$

Question 2.2

Write $\left|\psi^T_{\text{rec}}\right\rangle = \sum_{x,i,b} \alpha_{x,i,b} \left|x\right\rangle \otimes \left|i,b\right\rangle$. We will decompose the state into $n+1$ mutually orthogonal states, following the proof of Lemma 3.5:

- $\left|\psi_\emptyset\right\rangle = \sum_{\substack{x,i,b \\ x_i=\emptyset}} \alpha_{x,i,b} \left|x\right\rangle \otimes \left|i,b\right\rangle$
- $\left|\psi_y\right\rangle = \sum_{\substack{x,i,b \\ x_i=y}} \alpha_{x,i,b} \left|x\right\rangle \otimes \left|i,b\right\rangle$ for all $0 \leq y < n$

Then it holds that

$$\left|\psi^T_{\text{rec}}\right\rangle = \left|\psi_\emptyset\right\rangle + \sum_{y=0}^{n-1} \left|\psi_y\right\rangle .$$

Now we write, using triangle inequality:

$$\begin{aligned}
\left\|\Pi_{succeed} \left|\psi^T\right\rangle\right\| &= \left\|\Pi_{succeed}(S^{\otimes n} \otimes \mathrm{Id}) \left|\psi^T_{\text{rec}}\right\rangle\right\| \\
&\leq \left\|\Pi_{succeed}(S^{\otimes n} \otimes \mathrm{Id}) \left|\psi_\emptyset\right\rangle\right\| + \sum_{y=0}^{n-1} \left\|\Pi_{succeed}(S^{\otimes n} \otimes \mathrm{Id}) \left|\psi_y\right\rangle\right\|
\end{aligned}$$

We will now compute these terms separately using our results from the previous question.

- $\left|\psi_\emptyset\right\rangle$:

$$\begin{aligned}
\left\|\Pi_{succeed}(S^{\otimes n} \otimes \mathrm{Id}) \left|\psi_\emptyset\right\rangle\right\|^2 &= \left\|\Pi_{succeed}(S^{\otimes n} \otimes \mathrm{Id}) \sum_{\substack{x,i,b \\ x_i=\emptyset}} \alpha_{x,i,b} \left|x\right\rangle \otimes \left|i,b\right\rangle\right\|^2 \\
&= \sum_{\substack{x,i,b \\ x_i=\emptyset}} |\alpha_{x,i,b}|^2 \cdot \left\|\Pi_{succeed}(S^{\otimes n} \otimes \mathrm{Id}) \left|x\right\rangle \otimes \left|i,b\right\rangle\right\|^2 \\
&= \frac{1}{n} \sum_{\substack{x,i,b \\ x_i=\emptyset}} |\alpha_{x,i,b}|^2 \\
&= \frac{1}{n} \|\psi_\emptyset\|^2
\end{aligned}$$

The second equality follows because $\Pi_{succeed}(S^{\otimes n} \otimes \mathrm{Id})$ preserves orthogonality between states $|x\rangle \otimes |i,b\rangle$ with $x_i = \emptyset$.

- $|\psi_1\rangle$:

$$\begin{aligned}\left\|\Pi_{succeed}(S^{\otimes n} \otimes \mathrm{Id})\,|\psi_1\rangle\right\|^2 &= \left\|\Pi_{succeed}(S^{\otimes n} \otimes \mathrm{Id}) \sum_{\substack{x,i,b\\x_i=1}} \alpha_{x,i,b}\,|x\rangle \otimes |i,b\rangle\right\|^2 \\ &= \sum_{\substack{x,i,b\\x_i=1}} |\alpha_{x,i,b}|^2 \cdot \left\|\Pi_{succeed}(S^{\otimes n} \otimes \mathrm{Id})\,|x\rangle \otimes |i,b\rangle\right\|^2 \\ &= \frac{n-1}{n} \sum_{\substack{x,i,b\\x_i=1}} |\alpha_{x,i,b}|^2 \\ &= \frac{n-1}{n} \|\psi_1\|^2\end{aligned}$$

- $|\psi_y\rangle$ for $y \neq 1$:

$$\begin{aligned}\left\|\Pi_{succeed}(S^{\otimes n} \otimes \mathrm{Id})\,|\psi_y\rangle\right\|^2 &= \left\|\Pi_{succeed}(S^{\otimes n} \otimes \mathrm{Id}) \sum_{\substack{x,i,b\\x_i=y}} \alpha_{x,i,b}\,|x\rangle \otimes |i,b\rangle\right\|^2 \\ &= \sum_{\substack{x,i,b\\x_i=y}} |\alpha_{x,i,b}|^2 \cdot \left\|\Pi_{succeed}(S^{\otimes n} \otimes \mathrm{Id})\,|x\rangle \otimes |i,b\rangle\right\|^2 \\ &= \frac{1}{n^2} \sum_{\substack{x,i,b\\x_i=y}} |\alpha_{x,i,b}|^2 \\ &= \frac{1}{n^2} \|\psi_y\|^2\end{aligned}$$

We conclude that

$$\begin{aligned}\left\|\Pi_{succeed}\,|\psi^T\rangle\right\| &\leq \left\|\Pi_{succeed}(S^{\otimes n} \otimes \mathrm{Id})\,|\psi_\emptyset\rangle\right\| + \sum_{y=0}^{n-1} \left\|\Pi_{succeed}(S^{\otimes n} \otimes \mathrm{Id})\,|\psi_y\rangle\right\| \\ &\leq \frac{1}{\sqrt{n}}\,\||\psi_\emptyset\rangle\| + \frac{\sqrt{n-1}}{\sqrt{n}}\,\||\psi_1\rangle\| + \frac{1}{n}\sum_{y\neq 1} \||\psi_y\rangle\|\end{aligned}$$

We note that $\| |\psi_1\rangle \| = \left\| \Pi_{\text{succeed}} \left| \psi^T_{\text{rec}} \right\rangle \right\| \leq \left\| \Pi_{\text{rec}} \left| \psi^T_{\text{rec}} \right\rangle \right\| = \sqrt{\Delta_T}$. We can also use Cauchy-Schwarz on the remaining terms and conclude that

$$\left\| \Pi_{succeed} \left| \psi^T \right\rangle \right\| \leq \sqrt{\Delta_T} + O\left(\frac{1}{\sqrt{n}} \right).$$

Question 2.3

Any successful SEARCH quantum query algorithm must satisfy $\left\| \Pi_{\text{succeed}} \left| \psi^T \right\rangle \right\|^2 \geq \frac{2}{3} \implies \left\| \Pi_{\text{succeed}} \left| \psi^T \right\rangle \right\| \geq \frac{1}{2}$. From the previous question, this implies that it must hold:

$$\sqrt{\Delta_T} + O\left(\frac{1}{\sqrt{n}} \right) \geq \frac{1}{2}$$

From the lecture, this means that

$$T \cdot \sqrt{\frac{10}{n}} + O\left(\frac{1}{\sqrt{n}} \right) \geq \frac{1}{2} \implies T = \Omega(\sqrt{n}).$$

Problem 2

Question 1

A classical (deterministic) algorithm is to query the input at positions $1, 2, 3, \ldots$ until the same number appears twice. From the birthday bound, we know that the query complexity of this algorithm is $O(\sqrt{n})$ with high probability.

Question 2

Define C_t to be the event that there is a collision after t queries. Then

$$\begin{aligned}\Pr[C_t] &= \Pr[C_{t-1}] + \Pr[\text{collision at } t \mid \neg C_{t-1}] \\ &= \Delta_{t-1} + \frac{t-1}{n}.\end{aligned}$$

Where the last equality follows because the t^{th} query can collide with any of the $t-1$ distinct values with probability $\frac{t-1}{n}$.

Now by expanding the Δ_{t-1} term we get that

$$\Delta_t = \frac{t-1}{n} + \frac{t-2}{n} + \cdots + \frac{1}{n} = \frac{t(t-1)}{2n} = O\left(\frac{t^2}{n}\right).$$

Thus any classical algorithm that succeeds with at least constant probability must satisfy $t = \Omega(\sqrt{n})$.

Question 3

We will prove this by induction. Initially, the state $|\psi^0_{\text{rec}}\rangle$ is supported onto basis states $|x\rangle \otimes |i, b\rangle$ such that $x = \emptyset^n$. Thus the statement holds for $t = 0$.

We now show that if the statement holds for $t = k$, it also holds for $t = k+1$. Recall that

$$\left|\psi_{\text{rec}}^{k+1}\right\rangle = U_{k+1} R \left|\psi_{\text{rec}}^{k}\right\rangle .$$

Since U_{k+1} does not affect the support of the oracle register, we only need to consider R applied on $\left|\psi_{\text{rec}}^{k}\right\rangle$.

We have seen in the lecture that we can decompose $\left|\psi_{\text{rec}}^{k}\right\rangle$ into $|\psi_\emptyset\rangle$ which is the span of $|x\rangle \otimes |i, b\rangle$ for x such that $x_i = \emptyset$, and $|\psi_y\rangle$ where $x_i \in \{0, \ldots, n-1\}$. From the proposition of the lecture, we know that

$$R\,|\psi_\emptyset\rangle \text{ adds a uniformly random value to } x_i$$

$$R\,|\psi_y\rangle \text{ either keeps } x_i \text{ the same, resamples, or deletes it.}$$

Hence in both cases, the number of recorded values increases by at most 1. Thus $\left|\psi_{\text{rec}}^{k+1}\right\rangle$ is supported over $|x\rangle$ with at most $k+1$ non-$\emptyset$ values.

Question 4

We will define the operator Π that projects onto $\text{span}\{|x\rangle \otimes |i, b\rangle \mid x \text{ contains collision}\}$. Thus $\Delta_t = \|\Pi \,|\psi_{\text{rec}}^t\rangle\|^2$. From the lecture, we have seen that

$$\sqrt{\Delta_t} \leq \sqrt{\Delta_{t-1}} + \|\Pi R \underbrace{(\text{Id} - \Pi)\left|\psi_{\text{rec}}^{t-1}\right\rangle}_{\in \ker(\Pi)}\|$$

Claim. For any recording state $|\psi\rangle \in \ker(\Pi)$ with $t-1$ queries, we have

$$\|\Pi R\,|\psi\rangle\| \leq O\left(\frac{\sqrt{t-1}}{\sqrt{n}}\right) \||\psi\rangle\|$$

Proof. We will closely follow the proof of Lemma 3.5 from the lecture notes. Write

$$|\psi\rangle = \sum_{x,i,b} \alpha_{x,i,b}\,|x\rangle \otimes |i, b\rangle .$$

Since $|\psi\rangle$ is in the kernel of Π, it means that $\alpha_{x,i,b}$ is non-zero only for x with no collisions. Additionally, since $|\psi\rangle$ has $t-1$ recording queries, the number of non-$\emptyset$ entries in the $|x\rangle$ in the support of $|\psi\rangle$ is at most $t-1$.

We will decompose $|\psi\rangle$ into $n+1$ mutually orthogonal states:

- $|\psi_\emptyset\rangle = \sum_{\substack{x,i,b \\ x_i=\emptyset}} \alpha_{x,i,b} |x\rangle \otimes |i,b\rangle$
- $|\psi_y\rangle = \sum_{\substack{x,i,b \\ x_i=y}} \alpha_{x,i,b} |x\rangle \otimes |i,b\rangle$ for all $0 \leq y < n$.

Then

$$\|\Pi R |\psi\rangle\| \leq \|\Pi R |\psi_\emptyset\rangle\| + \sum_y \|\Pi R |\psi_y\rangle\|.$$

We bound each term separately.

- $$R|\psi_\emptyset\rangle = \frac{1}{\sqrt{n}} \sum_{\substack{x,i,b \\ x_i=\emptyset}} \alpha_{x,i,b} \sum_y \omega^{by} |\ldots, x_i = y, \ldots\rangle \otimes |i,b\rangle$$
  $$\implies \Pi R|\psi_\emptyset\rangle = \frac{1}{\sqrt{n}} \sum_{\substack{x,i,b \\ x_i=\emptyset}} \alpha_{x,i,b} \sum_{y \in \mathrm{supp}(x)} \omega^{by} |\ldots, x_i = y, \ldots\rangle \otimes |i,b\rangle$$
  $$\implies \|\Pi R|\psi_\emptyset\rangle\|^2 = \frac{1}{n} \sum_{\substack{x,i,b \\ x_i=\emptyset}} |\alpha_{x,i,b}|^2 \sum_{y \in \mathrm{supp}(x)} |\omega^{by}|^2 \leq \frac{t-1}{n} \||\psi_\emptyset\rangle\|^2.$$

  Where the last equality holds because the support of x is at most $t-1$ (since we only made $t-1$ queries).

- Following a similar proof we deduce that

  $$\|\Pi R|\psi_y\rangle\|^2 \leq \frac{9(t-1)}{n^2} \||\psi_y\rangle\|^2.$$

We conclude from the Cauchy-Schwarz inequality that

$$\|\Pi R|\psi\rangle\| \leq \frac{\sqrt{t-1}}{\sqrt{n}} \|\Pi R|\psi_\emptyset\rangle\| + \frac{3\sqrt{t-1}}{n} \sum_y \|\Pi R|\psi_y\rangle\| \leq O\left(\frac{\sqrt{t-1}}{\sqrt{n}}\right) \||\psi\rangle\|.$$

□

Thus we have proved that

$$\begin{aligned}\sqrt{\Delta_t} &\leq \sqrt{\Delta_{t-1}} + O\left(\frac{\sqrt{t-1}}{\sqrt{n}}\right)\\ &\leq O\left(\frac{\sqrt{t-1}}{\sqrt{n}}\right) + O\left(\frac{\sqrt{t-2}}{\sqrt{n}}\right) + \cdots + O\left(\frac{1}{\sqrt{n}}\right)\\ &= O\left(\frac{t^{3/2}}{\sqrt{n}}\right)\end{aligned}$$

Thus the probability that the record contains a collision after t quantum queries is $\Delta_t = O(t^3/n)$.

Note. This does not directly imply that a quantum query algorithm requires $\Omega(n^{1/3})$ quantum queries to solve the Collision problem since we have to argue that the progress is close to the success probability. This can be proven in a similar manner.

Problem 3

Question 1

Any deterministic query algorithm can keep track of the bipartite graph G and update the edges of the graph, maintaining the invariant that if edge (x, y) is in the graph, then the algorithm cannot distinguish between inputs x and y.

If the deterministic algorithm queries input bit i, then it can distinguish all pairs (x, y) that differ on coordinate i. Hence the edges of G_i can be removed from G.

Note that we originally have $|E|$ pairs, and each time we are removing the edges of G_i, which are $|E_i| \leq \max_i |E_i|$. Thus any deterministic algorithm that can distinguish all pairs in G, needs at least $\frac{|E|}{\max_i |E_i|} = \min_i \frac{|E|}{|E_i|}$ queries to the input.

We can deduce that $|E| \geq \max\{m_0|V_0|, m_1|V_1|\}$, and $|E_i| \leq \max\{\ell_{0,i}|V_0|, \ell_{1,i}|V_1|\}$. Hence

$$\min_i \frac{|E|}{|E_i|} \geq \frac{\max\{m_0|V_0|, m_1|V_1|\}}{\max_i\{\ell_{0,i}|V_0|, \ell_{1,i}|V_1|\}} \geq \min_i \left\{ \frac{m_0}{\ell_{0,i}} + \frac{m_1}{\ell_{1,i}} \right\}.$$

Question 2

We have seen in lecture that

$$Q(f) \geq \max_\Gamma \frac{\|\Gamma\|}{40 \cdot \max_i \|\Gamma_i\|}.$$

We will use the Γ given by

$$\Gamma_{x,y} = \mathbf{1}[(x, y) \in G].$$

One can verify that Γ_i also corresponds to the edges of G_i. Thus we want to show that

$$\|\Gamma\| \geq \Omega\left(\sqrt{m_0 m_1}\right)$$

and

$$\|\Gamma_i\| \leq O\left(\sqrt{\ell_0 \ell_1}\right) \ \forall\ i.$$

Bound $\|\Gamma\|$. We will use the definition of the spectral norm. But before that, observe that Γ is a block matrix of the form

$$\Gamma_i = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}.$$

Then we know that $\|\Gamma\| = \max\{\|A\|, \|B\|\}$. Now we can use the definition:

$$\|A\| = \max_x \frac{\|Ax\|}{\|x\|}.$$

Note that A is a $|V_0| \times |V_1|$ matrix and B a $|V_1| \times |V_0|$ matrix. Consider x to be the all-ones vector of length $|V_1|$ and y the all-ones vector of length $|V_0|$. Then

$$\frac{\|Ax\|}{\|x\|} \geq \frac{\sqrt{m_0^2 |V_0|}}{\sqrt{|V_1|}}, \qquad \frac{\|By\|}{\|y\|} \geq \frac{\sqrt{m_1^2 |V_1|}}{\sqrt{|V_0|}}$$

Thus

$$\|\Gamma\| \geq \max\left\{ \frac{m_0 \sqrt{|V_0|}}{\sqrt{|V_1|}}, \frac{m_1 \sqrt{|V_1|}}{\sqrt{|V_0|}} \right\} \geq \sqrt{\frac{m_0 \sqrt{|V_0|}}{\sqrt{|V_1|}} \cdot \frac{m_1 \sqrt{|V_1|}}{\sqrt{|V_0|}}} = \sqrt{m_0 m_1}.$$

Bound $\|\Gamma_i\|$. We will use the inequality given in the hint. But before that, observe that Γ_i is a block matrix of the form

$$\Gamma_i = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}.$$

Then we know that $\|\Gamma_i\| = \max\{\|A\|, \|B\|\}$. Now we can use the hint to get:

$$\|A\| \leq \max_{i,j} \|A_{i,\cdot}\| \cdot \|A_{\cdot,j}\|$$

$$\implies \|A\| \leq \sqrt{\ell_{0,i} \ell_{1,i}}.$$

The same holds for B, and thus we conclude that $\|\Gamma_i\| \leq \sqrt{\ell_{0,i} \ell_{1,i}}$.

Question 3

We will construct a bipartite graph over V_0, V_1 of the $f := k$-Threshold function and use the quantum adversary to lower bound the quantum query complexity of f.

We will define the edges of our graph to be $(x, y) \in V_0 \times V_1$, where $|x|= k-1$, $|y|= k$ and there exists a unique i such that $x_i = 0 \neq 1 = y_i$. In other words, for every input x with Hamming weight $k-1$, we flip each of its $n-k+1$ 0 bits to obtain the $n-k+1$ neighbors of x.

For every input y with Hamming weight k, we flip each of its k 1 bits to obtain its k neighbors in V_0. Thus $m_0 = n-k+1$ and $m_1 = k$.

Additionally, for every node $x \in V_0$ and every index i, there exists at most one neighbor $y \in V_1$. The same holds for all $y \in V_1$, by the construction of our graph. Thus $\ell_0 = \ell_1 = 1$.

From the results of Question 1, we conclude that

$$D(f) \geq \max\{m_0/\ell_0, m_1/\ell_1\} = \max\{n-k+1, k\}.$$

In the quantum case,

$$Q(f) \geq \sqrt{\frac{m_0 m_1}{\ell_0 \ell_1}} = \sqrt{k(n-k+1)}.$$

Question 4

Note. For this problem we will use a stronger version of the result we proved in Question 2. Let $\ell_{v,i}$ be the degree of vertex v in Γ_i ($v \in V_0 \cup V_1$). We define m_0, m_1 as in Question 2. Then

$$D(f) \geq \Omega\left(\min_i \min_{(x,y)\in\Gamma_i} \frac{m_0}{\ell_{x,i}} + \frac{m_1}{\ell_{y,i}}\right)$$

$$Q(f) \geq \Omega\left(\frac{\sqrt{m_0 m_1}}{\max_i \max_{(x,y)\in\Gamma_i} \sqrt{\ell_{x,i}\ell_{y,i}}}\right)$$

As per the hint, we will take

$$V_0 = \{x \in \{0,1\}^{\binom{n}{2}} : x \text{ represents two disjoint cycles, each of length} \geq \frac{n}{4}\}$$

$$V_1 = \{x \in \{0,1\}^{\binom{n}{2}} : x \text{ represents a cycle graph}\}$$

We will define our bipartite graph G to include edges $(x, y) \in V_0 \times V_1$ if the cycle graph y can be obtained by removing an edge from each of the two cycles and 'glue' the endpoints of the two paths of x.

As an example, if x consists of cycles C_1, C_2, we can remove the edges (a_1, b_1) and (a_2, b_2) respectively. Then we can obtain a cycle graph by connecting $a_1 - a_2$ and $b_1 - b_2$, or $a_1 - b_2$ and $a_2 - b_1$.

Bounding m_0, m_1. From the example above we can see that each pair of disjoint cycles can be made to a cycle in $\Theta(n^2)$ ways. This is because we need to choose an edge from each cycle and then there are two ways to glue the endpoints together. Since the cycles have length $\Omega(n)$, the total number of ways to remove the two edges is $\Theta(n^2)$.

Similarly, for the degree of a cycle $y \in V_1$, we can choose any edge e_1 of the cycle (n choices) and then another edge e_2 (which has to be sufficiently far in order for the disjoint cycles to be sufficiently long, but there are still $\Omega(n)$ edge choices). This implies that $m_1 \geq \Omega(n^2)$.

Bounding $\ell_{x,i}\ell_{y,i}$. We now consider an input x and a bit i (that corresponds to an edge). If $x_i = 1$, then edge i corresponds to an edge that was removed from one of the cycles to construct cycle y. Since the other edge to be removed can be any of $\Theta(n)$ edges, we conclude that $\ell_{x,i} = O(n)$. Now consider a neighboring cycle y of x. Since x, y are neighbors in Γ_i, this means that edge i is the edge that was added to y to 'close' one of the cycles. Let the endpoints of this edge be (a, b). Then we know that the edges removed from y must be one of the incident edges of a and b, thus giving at most a constant number of ways to break y into disjoint cycles. Thus $\ell_{y,i} = O(1)$.

If $x_i = 0$, then edge i is an edge that was added to glue together the two cycles. Say the endpoints of this edge are (a, b). Then the other added edge to x must be the edge that connects one of the two neighbors of a with the two neighbors of b. Thus there is only a constant number of ways to glue together the two cycles in Γ_i, hence $\ell_{x,i} = 1$ in this case. Since $x_i \neq y_i = 1$, then (a, b) is one of the edges that we remove to make the two cycles. The other edge is one of $O(n)$ edges of the cycle y, thus $\ell_{y,i} = O(n)$ in this case.

In conclusion, in both cases $\ell_{x,i}\ell_{y,i} = O(n)$ for all $(x, y) \in \Gamma_i$. Thus

$$Q(f) \geq \Omega\left(\sqrt{\frac{n^4}{n}}\right) = \Omega(n^{3/2}).$$

Also note that for each $(x, y) \in \Gamma_i$, at least one of $\ell_{x,i}, \ell_{y,i}$ is constant. Thus

$$D(f) \geq \Omega\left(\frac{n^2}{1} + \frac{n^2}{n}\right) = \Omega(n^2).$$