Proof sketches

1 Lecture 1

Lemma 1.1. $\||\psi^0_{\vec{0}}\rangle - |\psi^0_{\vec{i}}\rangle\| = 0$

Proof. $|\psi^0_{\vec{0}}\rangle = |\psi^0_{\vec{i}}\rangle = U_0|0,0\rangle$ □

Lemma 1.2. $\||\psi^T_{\vec{0}}\rangle - |\psi^T_{\vec{i}}\rangle\| \geq 1/3$ *if the algorithm succeeds wp* $\geq 2/3$ *after* T *queries*

Proof. Success conditions: $\|(\mathrm{Id} \otimes |0\rangle\langle 0|)|\psi^T_{\vec{0}}\rangle\|^2 \geq 2/3$ and $\|(\mathrm{Id} \otimes |1\rangle\langle 1|)|\psi^T_{\vec{i}}\rangle\|^2 \geq 2/3$

$$\begin{aligned}
\||\psi^T_{\vec{0}}\rangle - |\psi^T_{\vec{i}}\rangle\|^2 &= 2(1 - \mathrm{Re}(\langle\psi^T_{\vec{0}}|\psi^T_{\vec{i}}\rangle)) \\
&= 2(1 - \mathrm{Re}(\langle\psi^T_{\vec{0}}|\,(\mathrm{Id} \otimes |0\rangle\langle 0|)\,|\psi^T_{\vec{i}}\rangle) - \mathrm{Re}(\langle\psi^T_{\vec{0}}|\,(\mathrm{Id} \otimes |1\rangle\langle 1|)\,|\psi^T_{\vec{i}}\rangle)) \\
&\geq 2(1 - \|(\mathrm{Id} \otimes |0\rangle\langle 0|)\psi^T_{\vec{0}}\| \cdot \|(\mathrm{Id} \otimes |0\rangle\langle 0|)\psi^T_{\vec{i}}\| - \|(\mathrm{Id} \otimes |1\rangle\langle 1|)\psi^T_{\vec{0}}\| \cdot \|(\mathrm{Id} \otimes |1\rangle\langle 1|)\psi^T_{\vec{i}}\|) \\
&\qquad \text{by Cauchy-Schwarz inequality} \\
&\geq 2(1 - 2\sqrt{2}/3) \qquad \text{by success conditions} \\
&\geq 1/9
\end{aligned}$$

□

Lemma 1.3. $\||\psi^{t+1}_{\vec{0}}\rangle - |\psi^{t+1}_{\vec{i}}\rangle\| \leq \||\psi^t_{\vec{0}}\rangle - |\psi^t_{\vec{i}}\rangle\| + \sqrt{q^t_i}$

Proof.

$$\begin{aligned}
\||\psi^{t+1}_{\vec{0}}\rangle - |\psi^{t+1}_{\vec{i}}\rangle\| &= \|U_{t+1}|\psi^t_{\vec{0}}\rangle - U_{t+1}O_{\vec{i}}|\psi^t_{\vec{i}}\rangle\| && \text{by definition and } O_{\vec{0}} = \mathrm{Id} \\
&= \||\psi^t_{\vec{0}}\rangle - O_{\vec{i}}|\psi^t_{\vec{i}}\rangle\| && \text{unitary preserves norm} \\
&= \|O_{\vec{i}}(|\psi^t_{\vec{0}}\rangle - |\psi^t_{\vec{i}}\rangle) + (\mathrm{Id} - O_{\vec{i}})|\psi^t_{\vec{0}}\rangle\| \\
&\leq \|O_{\vec{i}}(|\psi^t_{\vec{0}}\rangle - |\psi^t_{\vec{i}}\rangle)\| + \|(\mathrm{Id} - O_{\vec{i}})|\psi^t_{\vec{0}}\rangle\| && \text{by triangle inequality} \\
&= \||\psi^t_{\vec{0}}\rangle - |\psi^t_{\vec{i}}\rangle\| + \|(\mathrm{Id} - O_{\vec{i}})|\psi^t_{\vec{0}}\rangle\|
\end{aligned}$$

We have $\mathrm{Id} - O_{\vec{i}} = |i\rangle\langle i| \otimes (\mathrm{Id} - X)$ where $X = |1\rangle\langle 0| + |0\rangle\langle 1|$. Hence, $\|(\mathrm{Id} - O_{\vec{i}})|\psi^t_{\vec{0}}\rangle\| = \|(|i\rangle\langle i| \otimes (\mathrm{Id} - X))|\psi^t_{\vec{0}}\rangle|\| \leq 2\|(|i\rangle\langle i| \otimes \mathrm{Id})|\psi^t_{\vec{0}}\rangle|\| = \sqrt{q^t_i}$, where we used that $\|\mathrm{Id} \otimes (\mathrm{Id} - X)\| \leq 2$. □

Theorem 1.4. $Q(\mathrm{OR}) \geq \sqrt{n}/3$

Proof. $n/3 \leq \sum_{i=1}^n \sum_{t=0}^T \sqrt{q^t_i} \leq \sqrt{nT \sum_{i=1}^n \sum_{t=0}^T q^t_i} = \sqrt{n}T \Rightarrow T \geq \sqrt{n}/3.$ □

2 Lecture 2

Proposition 2.1. *Fix a quantum algorithm making T queries. Let $p(x) \in [0,1]$ denote the probability that it outputs 1 on input x. Then $\deg(p) \le 2T$.*

Proof. By induction on T: for all $1 \le i \le n$, $b \in \{0,1\}$, $\langle i, b|\psi_x^T\rangle$ is a polynomial in x of degree $\le T$.

For $T = 0$, $|\psi_x^0\rangle = U_0|0,0\rangle$. Hence, $\langle i, b|\psi_x^0\rangle$ is independent from x.

For $T + 1$,

$$\begin{aligned}\langle i, b|\psi_x^{T+1}\rangle &= \langle i, b|\, U_{T+1} O_x \,|\psi_x^T\rangle \\ &= \sum_{j,c} \alpha_{j,c}\langle j, c|\, O_x \,|\psi_x^T\rangle \quad \text{where we define } \textstyle\sum_{j,c} \alpha_{j,c}^\dagger |j, c\rangle = U_{T+1}^\dagger |i, b\rangle \text{ (indep. from } x) \\ &= \sum_{j,c} \alpha_{j,c}((1 - x_j)\langle j, c|\psi_x^T\rangle + x_j\langle j, c \oplus 1|\psi_x^T\rangle) \\ &\qquad\qquad \text{since } O_x|j, c\rangle = |j, c \oplus x_j\rangle = (1 - x_j)|j, c\rangle + x_j|j, c \oplus 1\rangle\end{aligned}$$

The proposition follows since $p(x) = \|(\mathrm{Id} \otimes |1\rangle\langle 1|)|\psi_x^T\rangle\|^2 = \sum_{1\le i\le n} |\langle i, 1|\psi_x^T\rangle|^2$. □

Theorem 2.2. $Q(f) = \widetilde{\deg}(f)/2$

Proof. Suppose a quantum algorithm computes f with probability $\ge 2/3$ and makes T queries. Let $p(x)$ denote the denote the probability that it outputs 1 on input x. Then:

1. $\deg(p) \le 2T$ by Proposition 2.1
2. $p(x) \ge 2/3$ when $f(x) = 1$ and $p(x) \le 1/3$ when $f(x) = 0$, by success condition

In particular, $|p(x) - f(x)| \le 1/3$ for all x. Hence, $\widetilde{\deg}(f) \le \deg(p) \le 2T$. □

Lemma 2.3. *P_{sym} is a polynomial in k and $\deg(P_{\mathrm{sym}}) \le \deg(P)$.*

Proof. Let $S \subseteq \{1, \dots, n\}$ and consider the monomial $x_S = \prod_{i\in S} x_i$.

$$\mathbb{E}_{x\sim B_k}[x_S] = \begin{cases} 0 & \text{if } k < |S| \\ \frac{\binom{n-|S|}{k-|S|}}{\binom{n}{k}} = \frac{k(k-1)\cdots(k-|S|+1)}{n(n-1)\cdots(n-|S|+1)} & \text{otherwise}\end{cases}$$

This is a polynomial in k of degree $\le |S|$. □

Lemma 2.4. *$P_{\mathrm{sym}}(0) \in [0, 1/3]$ and $P_{\mathrm{sym}}(k) \in [2/3, 1]$ for $k \ge 1$.*

Proof. $P(x) \in [0, 1/3]$ for all $x \in B_0$ hence $P_{\mathrm{sym}}(0) = \mathbb{E}_{x\sim B_0}[P(x)] \in [0, 1/3]$.

Similarly, $P(x) \in [2/3, 1]$ for all $x \in B_k$, $k \ge 1$. □

Lemma 2.5. $\sum_x \phi(x) \cdot P(x) = 0,\ \forall P, \deg(P) < d \Leftrightarrow \phi$ *has no monomial of degree* $< d$.

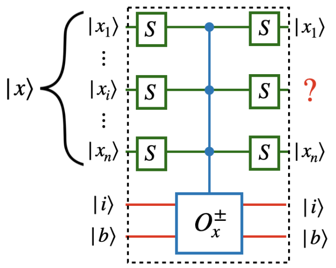
Proof. For any two subsets $S, T \subseteq \{1, \dots, n\}$, $\sum_{x\in\{-1,1\}^n} x_S x_T = \sum_{x\in\{-1,1\}^n} x_{S\cap T}^2 x_{T\setminus S} x_{S\setminus T} = \sum_{x\in\{-1,1\}^n} x_{T\setminus S} x_{S\setminus T} = 2^n \cdot \mathbf{1}_{S=T}$. □

3 Lecture 3

$$S:\begin{cases} |\varnothing\rangle & \mapsto \frac{1}{\sqrt{n}}\sum_{0\le y<n}|y\rangle \\ \frac{1}{\sqrt{n}}\sum_{0\le y<n}|y\rangle & \mapsto |\varnothing\rangle \\ \frac{1}{\sqrt{n}}\sum_{0\le y<n}\omega^{by}|y\rangle & \mapsto \frac{1}{\sqrt{n}}\sum_{0\le y<n}\omega^{by}|y\rangle \quad \text{if } 0<b<n \end{cases}$$

One can check that:

1. $S^{-1}=S^\dagger$ (unitary) and $S=S^\dagger$ (Hermitian)
2. For all $0\le y<n$, $S|y\rangle = |y\rangle + \frac{1}{\sqrt{n}}|\varnothing\rangle - \frac{1}{n}\sum_{0\le z<n}|z\rangle$



$R = S^{\otimes n}O^{\pm}S^{\otimes n}$

Lemma 3.1. $R|\dots,x_i=\varnothing,\dots\rangle\otimes|i,b\rangle = \frac{1}{\sqrt{n}}\sum_{0\le y<n}\omega^{by}|\dots,x_i=y,\dots\rangle\otimes|i,b\rangle$ *when* $b\neq 0$.

Proof. We focus on the i-th input register since it is the only register that can change upon applying R.

$$\begin{aligned} |x_i=\varnothing\rangle &\mapsto_S \frac{1}{\sqrt{n}}\sum_{0\le y<n}|y\rangle \\ &\mapsto_O \frac{1}{\sqrt{n}}\sum_{0\le y<n}\omega^{by}|y\rangle \\ &\mapsto_S \frac{1}{\sqrt{n}}\sum_{0\le y<n}\omega^{by}|y\rangle && \text{since } b\neq 0 \end{aligned}$$

If $b=0$ then R acts as the identity: $R|\dots,x_i=\varnothing,\dots\rangle\otimes|i,b\rangle = |\dots,x_i=\varnothing,\dots\rangle\otimes|i,b\rangle$. □

Lemma 3.2. $R|\dots,x_i=y,\dots\rangle\otimes|i,b\rangle = \omega^{by}|\dots,x_i=y,\dots\rangle\otimes|i,b\rangle + |\text{error}_y\rangle$ *when* $b\neq 0$

Proof.

$$\begin{aligned} |x_i=y\rangle &\mapsto_S |y\rangle + \frac{1}{\sqrt{n}}|\varnothing\rangle - \frac{1}{n}\sum_{0\le z<n}|z\rangle \\ &\mapsto_O \omega^{by}|y\rangle + \frac{1}{\sqrt{n}}|\varnothing\rangle - \frac{1}{n}\sum_{0\le z<n}\omega^{bz}|z\rangle \\ &\mapsto_S \omega^{by}|y\rangle + \frac{\omega^{by}}{\sqrt{n}}|\varnothing\rangle - \frac{1}{n}\sum_{0\le z<n}\omega^{by}|z\rangle + \frac{1}{n}\sum_{0\le z<n}|z\rangle - \frac{1}{n}\sum_{0\le z<n}\omega^{bz}|z\rangle \end{aligned}$$

□

Lemma 3.3. $\Delta_0 = 0$.

Proof. $|\psi^0_{\text{rec}}\rangle = |\varnothing,\dots,\varnothing\rangle\otimes|0,0\rangle$ □

Lemma 3.4. $\sqrt{\Delta_{t+1}} \leq \sqrt{\Delta_t} + \sqrt{10/n}$.

Proof.

$$\begin{aligned}
\sqrt{\Delta_{t+1}} &= \|\Pi U_{t+1} R|\psi^t_{\text{rec}}\rangle\| && \text{by } |\psi^{t+1}_{\text{rec}}\rangle = U_{t+1}R|\psi^t_{\text{rec}}\rangle \\
&= \|\Pi R|\psi^t_{\text{rec}}\rangle\| && \text{since } \Pi \text{ and } U_{t+1} \text{ commute} \\
&\leq \|\Pi R \Pi|\psi^t_{\text{rec}}\rangle\| + \|\Pi R(\text{Id}-\Pi)|\psi^t_{\text{rec}}\rangle\| && \text{by triangle inequality} \\
&\leq \|\Pi|\psi^t_{\text{rec}}\rangle\| + \|\Pi R(\text{Id}-\Pi)|\psi^t_{\text{rec}}\rangle\| && \text{by contraction} \\
&= \sqrt{\Delta_t} + \|\Pi R(\text{Id}-\Pi)|\psi^t_{\text{rec}}\rangle\|
\end{aligned}$$

Lemma 3.5. *For any state* $|\psi\rangle \in \ker(\Pi)$ *we have* $\|\Pi R|\psi\rangle\| \leq \sqrt{\frac{10}{n}}\||\psi\rangle\|$

Proof. Let $|\psi\rangle = \sum_{x,i,b} \alpha_{x,i,b}|x\rangle \otimes |i,b\rangle$ (by assumption, $\alpha_{x,i,b} \neq 0 \Rightarrow 1 \notin x$)

We decompose $|\psi\rangle$ into $n+2$ mutually orthogonal states:

- $|\psi_{\text{id}}\rangle = \sum_{x,i,b:b=0} \alpha_{x,i,b}|x\rangle \otimes |i,b\rangle$
- $|\psi_{\varnothing}\rangle = \sum_{x,i,b:x_i=\varnothing,b\neq 0} \alpha_{x,i,b}|x\rangle \otimes |i,b\rangle$
- $|\psi_y\rangle = \sum_{x,i,b:x_i=y,b\neq 0} \alpha_{x,i,b}|x\rangle \otimes |i,b\rangle$ for all $0 \leq y < n$

We show that:

- $\|\Pi R|\psi_{\text{id}}\rangle\| = 0$ since $R|\psi_{\text{id}}\rangle = |\psi_{\text{id}}\rangle$
- $\|\Pi R|\psi_{\varnothing}\rangle\| = \frac{1}{\sqrt{n}}\||\psi_{\varnothing}\rangle\|$
- $\|\Pi R|\psi_1\rangle\| = 0$ since $|\psi_1\rangle = 0$
- $\|\Pi R|\psi_y\rangle\| \leq \frac{3}{n}\||\psi_y\rangle\|$ for all $y \in \{0, 2, \ldots, n-1\}$

It will imply by triangle inequality + Cauchy-Schwarz:

$$\|\Pi R|\psi\rangle\| \leq \|\Pi R|\psi_{\varnothing}\rangle\| + \sum_y \|\Pi R|\psi_y\rangle\| \leq \frac{1}{\sqrt{n}}\||\psi_{\varnothing}\rangle\| + \frac{3}{n}\sum_{y\neq 1}\||\psi_y\rangle\| \leq \sqrt{\frac{10}{n}}\||\psi\rangle\|$$

Proof that $\|\Pi R|\psi_{\varnothing}\rangle\| \leq \frac{1}{\sqrt{n}}\||\psi_{\varnothing}\rangle\|$:

By Lemma 3.1, for any basis state $|x\rangle \otimes |i,b\rangle \in \text{supp}(|\psi_{\varnothing}\rangle)$ with $b \neq 0$, we have

$$\Pi R|x_1, \ldots, x_i = \varnothing, \ldots, x_n\rangle \otimes |i,b\rangle = \frac{\omega^b}{\sqrt{n}}|x_1, \ldots, 1, \ldots, x_n\rangle \otimes |i,b\rangle$$

Thus,

$$\Pi R|\psi_{\varnothing}\rangle = \sum_{x,i,b:x_i=\varnothing,b\neq 0} \alpha_{x,i,b}\frac{\omega^b}{\sqrt{n}}|x^{\{i\}}\rangle \otimes |i,b\rangle$$

where $x^{\{i\}}_i = 1 - x_i = 1$ and $x^{\{i\}}_j = x_j$ for $j \neq i$.

Finally, $\|\Pi R|\psi_{\varnothing}\rangle\|^2 = \sum_{x,i,b:x_i=\varnothing,b\neq 0} \frac{|\alpha_{x,i,b}|^2}{n} = \frac{1}{n}\||\psi_{\varnothing}\rangle\|^2$.

Proof that $\|\Pi R|\psi_y\rangle\| \leq \frac{3}{n}\||\psi_y\rangle\|$:

By Lemma 3.2,

$$\Pi R|x_1, \ldots, x_i = y, \ldots, x_n\rangle \otimes |i,b\rangle = \frac{1-\omega^b-\omega^{by}}{n}|x_1, \ldots, 1, \ldots, x_n\rangle \otimes |i,b\rangle$$

Hence, $\|\Pi R|\psi_y\rangle\|^2 = \sum_{x,i,b:x_i=y,b\neq 0} \frac{|(1-\omega^b-\omega^{by})\alpha_{x,i,b}|^2}{n^2} \leq \frac{9}{n^2}\||\psi_y\rangle\|^2$.

□

□

4 Lecture 4

Lemma 4.1. $\Delta_0 = \|\Gamma\|$.

Proof. $\Delta_0 = |\langle\psi^0|\,\Gamma\otimes\mathrm{Id}\,|\psi^0\rangle| = |\langle a|\,\Gamma\,|a\rangle| = \|\Gamma\|$ since $|\psi^0\rangle = |a\rangle\otimes|0,0\rangle$ and a is a principal ev of unit norm. □

Lemma 4.2. $\Delta_T < 0.95\|\Gamma\|$ *if the algorithm succeeds wp* $\geq 2/3$ *after* T *queries.*

Proof. Define $\Pi_{\text{succeed}} = \sum_x |x\rangle\langle x|\otimes\mathrm{Id}\otimes|f(x)\rangle\langle f(x)|$

$$\begin{aligned}
\Delta_T &= |\langle\psi^T|\,\Gamma\otimes\mathrm{Id}\,|\psi^T\rangle| \\
&= \big|\langle\psi^T|\,\Pi_{\text{succeed}}(\Gamma\otimes\mathrm{Id})\Pi_{\text{succeed}}\,|\psi^T\rangle + \langle\psi^T|\,(\mathrm{Id}-\Pi_{\text{succeed}})(\Gamma\otimes\mathrm{Id})(\mathrm{Id}-\Pi_{\text{succeed}})\,|\psi^T\rangle \\
&\quad + \langle\psi^T|\,\Pi_{\text{succeed}}(\Gamma\otimes\mathrm{Id})(\mathrm{Id}-\Pi_{\text{succeed}})\,|\psi^T\rangle + \langle\psi^T|\,(\mathrm{Id}-\Pi_{\text{succeed}})(\Gamma\otimes\mathrm{Id})\Pi_{\text{succeed}}\,|\psi^T\rangle\big| \\
&= |\langle\psi^T|\,\Pi_{\text{succeed}}(\Gamma\otimes\mathrm{Id})(\mathrm{Id}-\Pi_{\text{succeed}})\,|\psi^T\rangle + \langle\psi^T|\,(\mathrm{Id}-\Pi_{\text{succeed}})(\Gamma\otimes\mathrm{Id})\Pi_{\text{succeed}}\,|\psi^T\rangle| \\
&\leq 2\|\Gamma\otimes\mathrm{Id}\|\cdot\|\Pi_{\text{succeed}}|\psi^T\rangle\|\cdot\|(\mathrm{Id}-\Pi_{\text{succeed}})|\psi^T\rangle\| \\
&\leq 2\|\Gamma\|\cdot\sqrt{\|\Pi_{\text{succeed}}|\psi^T\rangle\|^2(1-\|\Pi_{\text{succeed}}|\psi^T\rangle\|^2)} \\
&\leq 2\|\Gamma\|\cdot\sqrt{2}/3
\end{aligned}$$

where the third equality is because $\Gamma_{xy} = 0$ when $f(x) = f(y)$. □

Lemma 4.3. $\Delta_{t+1} \geq \Delta_t - 2\max_{1\leq i\leq n}\|\Gamma_i\|$.

Proof.

$$\begin{aligned}
\Delta_t - \Delta_{t+1} &\leq |\langle\psi^t|\,\Gamma\otimes\mathrm{Id}\,|\psi^t\rangle - \langle\psi^{t+1}|\,\Gamma\otimes\mathrm{Id}\,|\psi^{t+1}\rangle| \\
&= |\langle\psi^t|\,\Gamma\otimes\mathrm{Id}\,|\psi^t\rangle - \langle\psi^t|\,O(\Gamma\otimes\mathrm{Id})O\,|\psi^t\rangle| \qquad \text{since } |\psi^{t+1}\rangle = (\mathrm{Id}\otimes U_{t+1})O|\psi^t\rangle \\
&= |\sum_{x,y}\Gamma_{x,y}a_x a_y^*\langle\psi_x^t|(\mathrm{Id}-O_xO_y)|\psi_y^t\rangle| \\
&= |\sum_i\sum_{x,y}(\Gamma_i)_{x,y}a_x a_y^*\langle\psi_x^t|(|i\rangle\langle i|\otimes(\mathrm{Id}-X))|\psi_y^t\rangle| \\
&\qquad \text{by Claim 4.4 and } (\Gamma_i)_{x,y} = \Gamma_{x,y}\cdot\mathbf{1}_{x_i\neq y_i} \\
&= |\sum_i\langle\psi^t|\,(\Gamma_i\otimes|i\rangle\langle i|\otimes(\mathrm{Id}-X))\,|\psi^t\rangle| \qquad \text{by } |\psi^t\rangle = \textstyle\sum_x a_x|x\rangle\otimes|\psi_x^t\rangle \\
&\leq \max_i\|\Gamma_i\otimes\mathrm{Id}\otimes(\mathrm{Id}-X)\|\cdot\sum_i\|(\mathrm{Id}\otimes|i\rangle\langle i|\otimes\mathrm{Id})|\psi^t\rangle\|^2 \\
&\qquad \text{by Cauchy-Schwarz inequality} \\
&= 2\max_i\|\Gamma_i\|
\end{aligned}$$

Claim 4.4. $\mathrm{Id} - O_xO_y = \sum_{i:x_i\neq x_j}|i\rangle\langle i|\otimes(\mathrm{Id}-X)$

Proof.

$$\langle i,b|(\mathrm{Id}-O_xO_y)|j,c\rangle = \begin{cases} 0 & \text{if } i\neq j \text{ or } x_i = x_j \\ 1 & \text{if } i = j,\ x_i\neq x_j,\ b = c \\ -1 & \text{if } i = j,\ x_i\neq x_j,\ b\neq c \end{cases}$$

□

□

5 Lecture 5

Claim 5.1. $\langle t_y^+ | t_x^- \rangle = \mathbf{1}_{f(x)=f(y)}$

Proof. $\langle t_y^+ | t_x^- \rangle = 1 - \sum_i \langle \bar{y}_i | x_i \rangle \langle w^{(y,i)} | w^{(x,i)} \rangle = 1 - \sum_{i: x_i \neq y_i} \langle w^{(y,i)} | w^{(x,i)} \rangle = 1 - \mathbf{1}_{f(x) \neq f(y)}$. □

Corollary 5.2. *If $f(x) = 0$ then $\Delta | t_x^- \rangle = 0$ and $\Pi_x | t_x^- \rangle = | \star \rangle$.*

Claim 5.3. *If $f(x) = 1$ then $U_x | t_x^+ \rangle = | t_x^+ \rangle$ (i.e. $P_0 | t_x^+ \rangle = | t_x^+ \rangle$).*

Proof. $U_x | t_x^+ \rangle = (2\Pi_x - \mathrm{Id})(2\Delta - \mathrm{Id}) | t_x^+ \rangle = (2\Pi_x - \mathrm{Id}) | t_x^+ \rangle = | t_x^+ \rangle$ □

Lemma 5.4. *If $f(x) = 1$ then $| \star \rangle = P_0 | \star \rangle + | \mathrm{err}_1 \rangle$ where $\| | \mathrm{err}_1 \rangle \|^2 \leq 1/3$.*

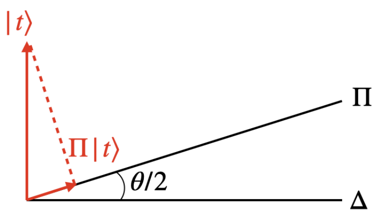
Proof.

$$
\begin{aligned}
\| | \mathrm{err}_1 \rangle \|^2 &= \| (\mathrm{Id} - P_0) | \star \rangle \|^2 \\
&= \frac{1}{3T} \| (\mathrm{Id} - P_0) \sum_i | i, x_i \rangle \otimes | w^{(x,i)} \rangle \|^2 \\
&\quad \text{since } | \star \rangle = | t_x^+ \rangle - \tfrac{1}{\sqrt{3T}} \textstyle\sum_i | i, x_i \rangle \otimes | w^{(x,i)} \rangle \text{ and } P_0 | t_x^+ \rangle = | t_x^+ \rangle \\
&\leq \frac{1}{3T} \| \sum_i | i, x_i \rangle \otimes | w^{(x,i)} \rangle \|^2 \\
&= \frac{1}{3T} \sum_i \| w^{(x,i)} \|^2 \\
&\leq 1/3
\end{aligned}
$$

□

Lemma 5.5. *If $f(x) = 0$ then $| \star \rangle = (\mathrm{Id} - P_{1/(2T)}) | \star \rangle + | \mathrm{err}_0 \rangle$ where $\| | \mathrm{err}_0 \rangle \|^2 \leq 1/3$.*

Claim 5.6 (Effective spectral gap lemma (in 2 dimensions))**.** $\| \Pi | t \rangle \| = \sin(\theta/2) \| | t \rangle \|$



Proof. □

Proof of Lemma 5.5.

$$
\begin{aligned}
\| | \mathrm{err}_0 \rangle \|^2 &= \| P_{1/(2T)} | \star \rangle \|^2 \\
&= \| P_{1/(2T)} \Pi_x | t_x^- \rangle \|^2 \\
&\leq \sin^2(1/(4T)) \| | t_x^- \rangle \|^2 \\
&\leq \frac{1}{16T^2} (1 + 3T \sum_i \| w^{(x,i)} \|^2) \\
&\leq \frac{1}{16T^2} (1 + 3T^2) \\
&\leq 1/3
\end{aligned}
$$

□