Quantum query complexity

Lecture 4

The adversary method

Materials: https://yassine-hamoudi.github.io/pcmi2023/

Focus of this lecture

The (generalized) adversary method

- A lower bound method that is always optimal

  > We'll show in lecture 5 how to turn it into an algorithm

  > Counterpart: often harder to use

- It shares some ideas with the hybrid method (lecture 1) and the recording method (lecture 3)

  (in fact: these can be seen as particular cases of it)

Reminders

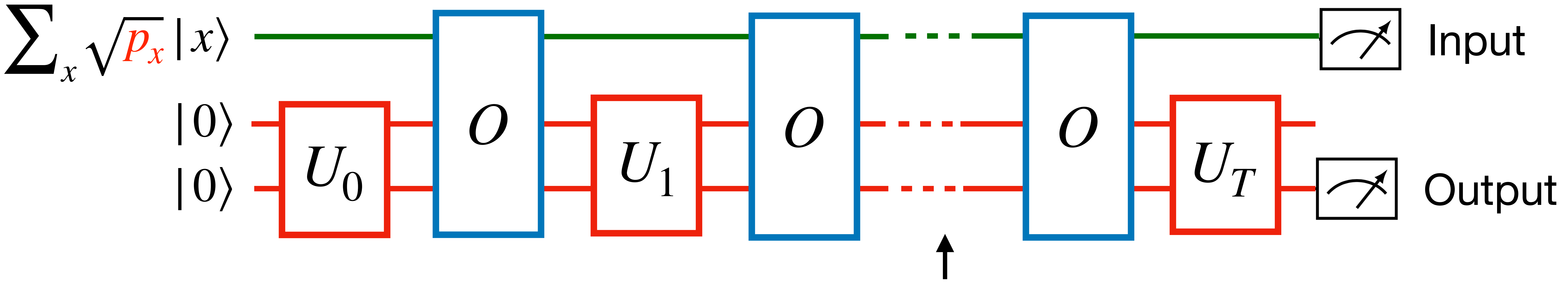
The distinguishing lemma (lecture 1)

The states $|\psi_x^T\rangle$ and $|\psi_y^T\rangle$ can be distinguished with probability $\geq 2/3$

if an only if there are “sufficiently orthogonal” $|\langle\psi_x^T|\psi_y^T\rangle| \leq 2\sqrt{2}/3$

The purification viewpoint (lecture 3)

We can set a distribution $(p_x)_x$ on the input by adding a purification register

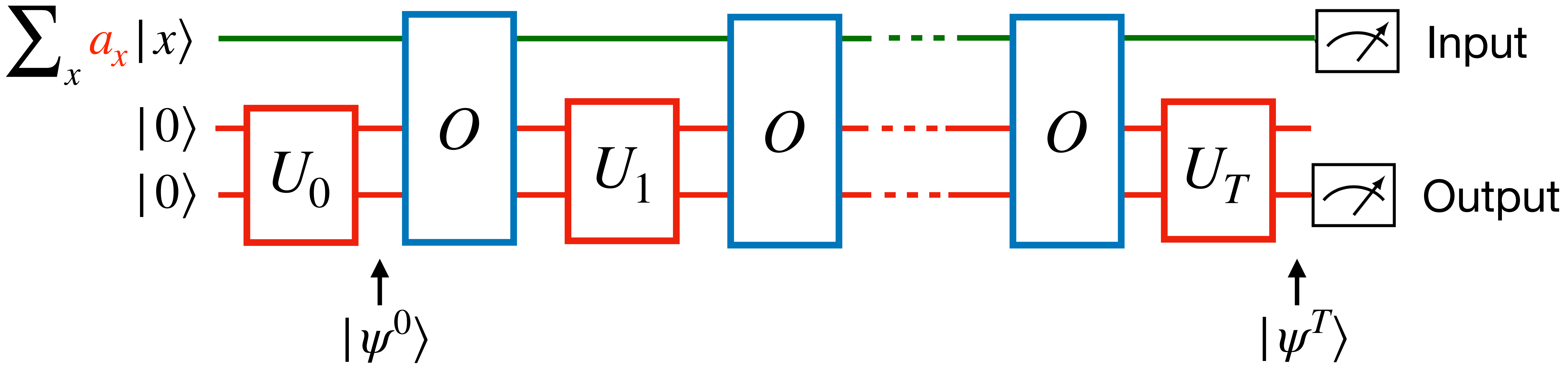


$|\psi^t\rangle = \sum_x \sqrt{p_x}|x\rangle \otimes |\psi_x^t\rangle$

Quantum adversary

First step: replace $(p_x)_x$ with complex numbers $(a_x)_x$ s.t. $\sum_x |a_x|^2 = 1$

$$|\psi^t\rangle = \sum_x a_x |x\rangle \otimes |\psi_x^t\rangle$$



First step: replace $(p_x)_x$ with complex numbers $(a_x)_x$ s.t. $\sum_x |a_x|^2 = 1$

$$|\psi^t\rangle = \sum_x a_x |x\rangle \otimes |\psi_x^t\rangle$$

Second step: consider the Gram matrix:

$$\sum_{x,y} a_x^* a_y \langle \psi_x^t | \psi_y^t \rangle \, |x\rangle\langle y|$$

$$\begin{array}{cc} & y \\ x & \left(\begin{array}{c} a_x^* a_y \langle \psi_x^t | \psi_y^t \rangle \end{array} \right) \end{array}$$

Third step: place some weights $\Gamma_{x,y}$ on the "hard" pairs of inputs

(symmetric) $\Gamma_{x,y} = \Gamma_{y,x}$

(consistent) if $f(x) = f(y)$ then $\Gamma_{x,y} = 0$

$$\begin{array}{cc} & y \\ x & \left(\begin{array}{c} \Gamma_{x,y} a_x^* a_y \langle \psi_x^t | \psi_y^t \rangle \end{array} \right) \end{array}$$

Adversary matrix: $\Gamma \in \mathbb{R}^{2^n \times 2^n}$ symmetric and $f(x) = f(y) \Rightarrow \Gamma_{x,y} = 0$

Adversary distribution: $a \in \mathbb{C}^{2^n}$ principal (unit) eigenvector of Γ

"Punctured" matrices: $\Gamma_i \in \mathbb{R}^{2^n \times 2^n}$ such that $(\Gamma_i)_{x,y} = \Gamma_{x,y} \cdot \mathbf{1}_{x_i \neq y_i}$

Progress measure: $\Delta_t = |\langle \psi^t | (\Gamma \otimes \mathrm{Id}) | \psi^t \rangle| = \left| \sum_{x,y} \Gamma_{x,y} a_x^* a_y \langle \psi_x^t | \psi_y^t \rangle \right|$

<u>Lemma 1:</u> $\Delta_0 = \|\Gamma\|$ *(initial condition)*

<u>Lemma 2:</u> $\Delta_T < 0.95 \|\Gamma\|$ if the algorithm succeeds wp $\geq 2/3$ *(final condition)*

<u>Lemma 3:</u> $\Delta_{t+1} \geq \Delta_t - 2 \max_{1 \leq i \leq n} \|\Gamma_i\|$ *(evolution)*

Theorem: $Q(f) \geq \max_{\Gamma} \dfrac{||\Gamma||}{40 \cdot \max_{1 \leq i \leq n} ||\Gamma_i||}$

- Positive-weight adversary: $\forall x, y, \Gamma_{x,y} \geq 0$
  - Has a nice combinatorial interpretation (see problem session)
  - Sub-optimal (the "certificate" and "property testing" barriers)

- Negative-weight adversary: $\forall x, y, \Gamma_{x,y} \in \mathbb{R}$
  - Optimal! (see next lecture) $Q(f) = \Theta\left(\max_{\Gamma} \dfrac{||\Gamma||}{\max_{1 \leq i \leq n} ||\Gamma_i||}\right)$

Applications

<u>OR function:</u> $f(x) = 0$ if and only if $x = (0,0,\dots,0)$

We (again) only focus on the $n + 1$ "hardest" inputs denoted by:

$\vec{0} = (0,0,\dots,0)$ $\quad$ $\vec{1} = (1,0,\dots,0)$ $\quad$ $\vec{2} = (0,1,0,\dots,0)$ $\quad \dots \quad$ $\vec{n} = (0,0,\dots,1)$

$$\Gamma = \begin{array}{cc} \begin{array}{cccc} \vec{0} & \vec{1} & \dots & \vec{n} \end{array} & \\ \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} & \begin{array}{c} \vec{0} \\ \vec{1} \\ \vdots \\ \vec{n} \end{array} \end{array} \qquad \Gamma_i = \begin{array}{cc} \begin{array}{ccccccc} & & & \vec{i} & & & \end{array} & \\ \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \vdots & & & & & \vdots \\ 1 & \vdots & & & & & \vdots \\ 0 & \vdots & & & & & \vdots \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 \end{pmatrix} & \begin{array}{c} \\ \\ \\ \vec{i} \\ \\ \\ \\ \end{array} \end{array}$$

(omitting the other 0-entries)

$\|\Gamma\| = \sqrt{n}$ $\qquad$ $\|\Gamma_i\| = 1$ $\qquad$ $\Rightarrow Q(\mathrm{OR}) \geq \sqrt{n}/40$