Communication Complexity

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Two player model

$$F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$$

Alice

$x \in \{0,1\}^n$

$F(x,y) = ?$

$$\xleftrightarrow[\text{channel}]{0,1}$$

Bob

$y \in \{0,1\}^n$

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$F(x,y) = ?$

Number of bits communicated?

- $\mathrm{D}_2(F)$: cost of the most efficient deterministic protocol
- $\mathrm{R}_2(F)$: cost of the most efficient randomized protocol with error 1/3

Two player simultaneous model

Alice

$x \in \{0,1\}^n$

Bob

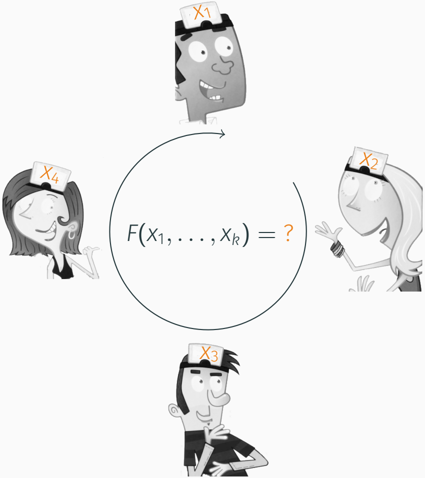
$y \in \{0,1\}^n$

Referee

$F(x,y) = ?$

Simultaneous communication complexity: $D_2^{||}(F)$ and $R_2^{||}(F)$

Number On the Forehead model



NOF model:

- Player i does not see x_i. Communicate by broadcasting
- Communication cost: $\mathrm{D}_k(F)$, $\mathrm{R}_k(F)$, $\mathrm{D}_k^{||}(F)$ and $\mathrm{R}_k^{||}(F)$

Applications of Communication Complexity

Circuit complexity [HG91, BT94]

Ramsey theory [CFL83]

Branching programs [CFL83]

Proof complexity [BPS07]

Quasirandom graphs [CT93]

Property testing [BBM12]

Streaming algorithms [AMS96]

Game theory [CS04, NS06]

Data structures [MNSW95]

Contents

The $\log n$ barrier and composed functions

The $\log n$ barrier:

Find a function F such that $\mathrm{D}_k^{||}(F) = \omega(\operatorname{polylog} n)$ when $k = \operatorname{polylog} n$.

ACC^0 and the $\log n$ barrier

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Find a function F such that $\mathsf{D}_k^{||}(F) = \omega(\text{polylog}\, n)$ when $k = \text{polylog}\, n$.

Motivations:

- ACC^0 = functions computable by polysize constant-depth circuits made of AND, OR, NOT and MOD_m gates
- NEXP $\not\subseteq$ ACC^0 [Wil14]
- Conjecture: NP $\not\subseteq$ ACC^0

ACC^0 and the $\log n$ barrier

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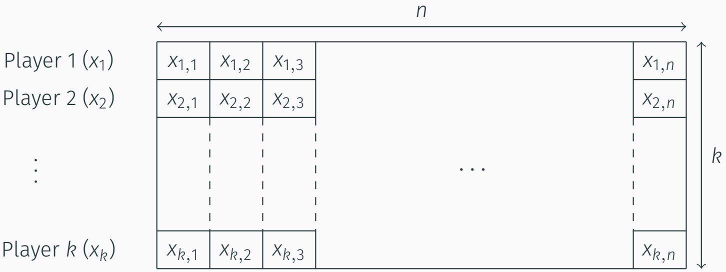
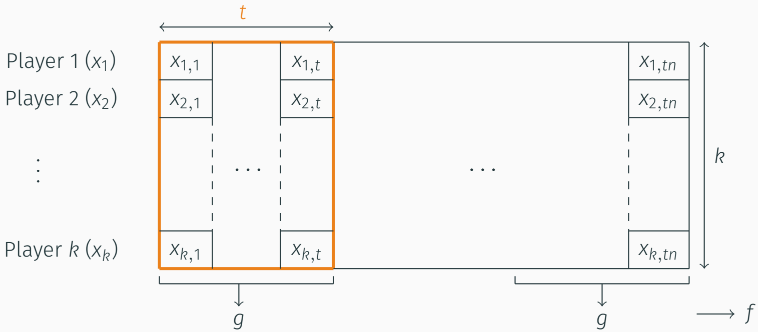
$$F \text{ breaks the } \log n \text{ barrier} \xrightarrow{\text{[HG91]}} F \notin \mathsf{ACC}^0$$


n
Player 1 (x_1)
Player 2 (x_2)
$\vdots$
Player k (x_k)
$x_{1,1}$ $x_{1,2}$ $x_{1,3}$ $x_{1,n}$
$x_{2,1}$ $x_{2,2}$ $x_{2,3}$ $x_{2,n}$
$\cdots$
$x_{k,1}$ $x_{k,2}$ $x_{k,3}$ $x_{k,n}$
k

Composed functions



Given $f : \{0,1\}^{n/t} \to \{0,1\}$ and $g : \{0,1\}^{t \cdot k} \to \{0,1\}$:

$$f \circ g(x_1, \ldots, x_k)$$

Prior work

Symmetric function = invariant under any permutation of the input

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When $k = \mathrm{polylog}\, n$:

- $\mathrm{D}_k(\textsc{Sym} \circ \mathrm{AND}_1) = \mathcal{O}\left(\log^2 n\right)$ [Gro94]
- $\mathrm{D}_k^{||}(\textsc{Sym} \circ \textsc{Sym}_1) = \mathcal{O}\left(\log^3 n\right)$ [BGKL04]
- $\mathrm{D}_k^{||}(\textsc{Sym} \circ \textsc{Any}_1) = \mathcal{O}\left(\log^3 n\right)$ [ACFN15]
- $\mathrm{D}_k(\textsc{Sym} \circ \textsc{Any}_t) = \mathcal{O}\left(polylogn\right)$ for $t \leq \log\log n$ [CS14]

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- $D_k^{||}(\textsc{Sym} \circ \textsc{Sym}_1) = \mathcal{O}\left(\log^3 n\right)$ [BGKL04]
- $D_k^{||}(\textsc{Sym} \circ \textsc{Any}_1) = \mathcal{O}\left(\log^3 n\right)$ [ACFN15]
- $D_k(\textsc{Sym} \circ \textsc{Any}_t) = \mathcal{O}\left(polylogn\right)$ for $t \leq \log\log n$ [CS14]

Our result:

- $D_k^{||}(\textsc{Sym} \circ \textsc{Sym}_t) = \mathcal{O}\left(polylogn\right)$ for constant t

Proof sketch

Symmetric f and g with $t = 2$:

0	0	0	0	0	1	1	0	0	0	1	0	1	1	1	0	0	1	0	1
0	1	0	0	1	1	0	1	0	0	0	1	0	1	0	0	1	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	1	1	0	0	1	0	0
0	0	0	0	1	0	0	1	0	0	0	1	1	0	0	1	1	0	0	0
1	1	0	0	0	0	1	1	0	1	0	0	0	1	0	0	1	0	0	0

g g $\cdots$ g $\longrightarrow f$

Proof sketch

Symmetric f and g with $t = 2$:

0	0	1	2	0	2	3	2	1	1
1	0	3	1	0	1	1	0	2	0
0	0	3	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
3	0	0	3	1	0	1	0	2	0

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Proof sketch

Symmetric f and g with $t = 2$:

0	0	1	2	0	2	3	2	1	1
1	0	3	1	0	1	1	0	2	0
0	0	3	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
3	0	0	3	1	0	1	0	2	0

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y_{i_1,i_2,i_3} = # columns with exactly i_1 occurrences of 1, i_2 of 2 and i_3 of 3

Proof sketch

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0	0	1	2	0	2	3	2	1	1
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y_{i_1,i_2,i_3} = # columns with exactly i_1 occurrences of 1, i_2 of 2 and i_3 of 3

$\rightarrow y_{0,0,0} = 1$

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Symmetric f and g with $t = 2$:

0	0	1	2	0	2	3	2	1	1
1	0	3	1	0	1	1	0	2	0
0	0	3	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
3	0	0	3	1	0	1	0	2	0

$g \quad g \quad \ldots \quad g \longrightarrow f$

y_{i_1,i_2,i_3} = # columns with exactly i_1 occurrences of 1, i_2 of 2 and i_3 of 3

$\rightarrow y_{0,0,0} = 1, y_{1,0,0} = 2$

Proof sketch

Symmetric f and g with $t = 2$:

0	0	1	2	0	2	3	2	1	1
1	0	3	1	0	1	1	0	2	0
0	0	3	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
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0	0	1	2	0	2	3	2	1	1
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$\rightarrow y_{0,0,0} = 1, y_{1,0,0} = 2, y_{0,4,1} = 0, \dots$

$\boxed{\text{Recovering the } y_{i_1,i_2,i_3}\text{'s is enough since } f \text{ and } g \text{ are symmetric}}$

Proof sketch

~~0~~	~~0~~	~~1~~	~~2~~	~~0~~	~~2~~	~~3~~	~~2~~	~~1~~	~~1~~
1	0	3	1	0	1	1	0	2	0
0	0	3	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
3	0	0	3	1	0	1	0	2	0

- Player 1 sends to the referee:

  $a^1_{i_1,i_2,i_3}$ = # columns he sees with i_1 occurrences of 1, i_2 of 2 and i_3 of 3

  $\rightarrow a^1_{0,0,0} = 2, a^1_{1,0,0} = 1, a^1_{2,1,1} = 1, \ldots$

Proof sketch

0	0	1	2	0	2	3	2	1	1
1	0	3	1	0	1	1	0	2	0
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  $\rightarrow a^1_{0,0,0} = 2, a^1_{1,0,0} = 1, a^1_{2,1,1} = 1, \ldots$
- Players 2 to 5 do the same

Proof sketch

The referee computes:

$$b_{i_1,i_2,i_3} = a^1_{i_1,i_2,i_3} + \cdots + a^5_{i_1,i_2,i_3}$$

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It verifies:

$$\begin{cases} y_{i_1,i_2,i_3} \geq 0 \\ \\ \\ \\ \end{cases}$$

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It verifies:

$$\begin{cases} y_{i_1,i_2,i_3} \geq 0 \\ \sum y_{i_1,i_2,i_3} = n \\ (k - (i_1 + i_2 + i_3))y_{i_1,i_2,i_3} + (i_1 + 1)y_{i_1+1,i_2,i_3} \\ \qquad + (i_2 + 1)y_{i_1,i_2+1,i_3} + (i_3 + 1)y_{i_1,i_2,i_3+1} = b_{i_1,i_2,i_3} \end{cases}$$

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Theorem

If $k \geq 5^{2^t} \log n$ then it admits exactly one integral solution.

$\rightarrow$ the referee recovers the y_{i_1,i_2,i_3}'s and computes the output

Decision tree complexity and log-rank conjecture

$$F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$$

Proposition ([MS82])

Let $M_F \in \{0,1\}^{n \times n}$ be the communication matrix: $M_F(x,y) = F(x,y)$.

$$\log \operatorname{rank} M_F \leq D_2(F)$$

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Conjecture

For some absolute constant c:

$$\log \operatorname{rank} M_F \leq D_2(F) \leq \log^c \operatorname{rank} M_F$$

XOR and AND functions

- A function $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ is an XOR function if:

$$F(x, y) = f(x \oplus y)$$

  for some $f : \{0,1\}^n \to \{0,1\}$

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$$F(x,y) = f(x \wedge y)$$

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Examples: $\textsc{Equality}(x,y) = \mathrm{NOR}(x \oplus y)$, $\textsc{Hamming}_d(x,y) = \mathrm{GAP}_d(x \oplus y)$, $\textsc{Disjointness}(x,y) = \mathrm{NOR}(x \wedge y)$, $\textsc{InnerProduct}(x,y) = \mathrm{MOD}_2(x \wedge y)$, etc.

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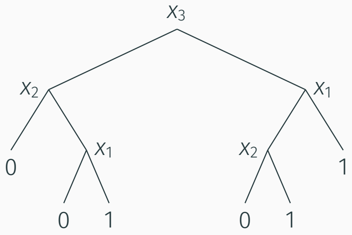
Interests:

- For XOR functions: $\operatorname{rank} M_F = \operatorname{mon} f$ [BC99]
- For AND functions: $\operatorname{rank} M_F = \operatorname{mon}^\star f$ [BdW01]
- Connections with Decision Tree complexity

A decision tree is an ordered tree where each internal node is labeled with a query, and each leaf is labeled with 0 or 1.



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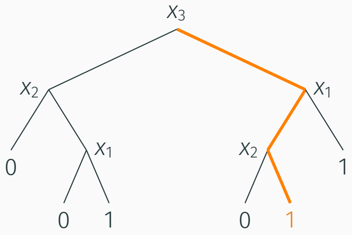

Input: $x_1x_2x_3 = 011$ on a regular decision tree

$$\mathrm{DT}(f), \mathrm{RDT}(f) \text{ and } \mathrm{QDT}(f)$$

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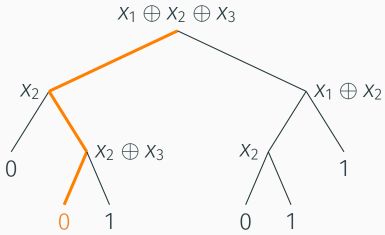

Input: $x_1x_2x_3 = 011$ on a parity decision tree

$$\mathrm{DT}^{\oplus}(f), \mathrm{RDT}^{\oplus}(f) \text{ and } \mathrm{QDT}^{\oplus}(f)$$

Decision tree complexity

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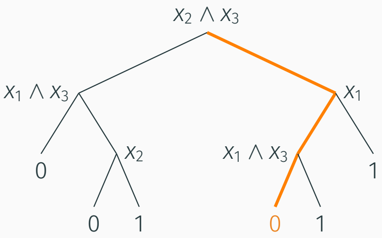

Input: $x_1x_2x_3 = 011$ on a conjunctive decision tree

$$\mathrm{DT}^{\wedge}(f), \mathrm{RDT}^{\wedge}(f) \text{ and } \mathrm{QDT}^{\wedge}(f)$$

Connections

Proposition ([ZS10])

For any XOR function $F(x, y) = f(x \oplus y)$:

$$D_2(F) \leq 2 \cdot DT^{\oplus}(f)$$

For any AND function $F(x, y) = f(x \wedge y)$:

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Connections

Proposition ([ZS10])

For any XOR function $F(x, y) = f(x \oplus y)$*:*

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Conjecture

- *Communication and Decision Tree complexities are polynomially related*

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For any AND function $F(x, y) = f(x \wedge y)$*:*

$$D_2(F) \leq 2 \cdot DT^{\wedge}(f)$$

Conjecture

- *Communication and Decision Tree complexities are polynomially related*
- *Log-rank conjecture for decision trees:*
  - *XOR function:* $\log \mathrm{mon}(f) \leq D_2(F) \leq 2 \cdot DT^{\oplus}(f) \leq \log^c \mathrm{mon}(f)$
  - *AND function:* $\log \mathrm{mon}^{\star}(f) \leq D_2(F) \leq 2 \cdot DT^{\wedge}(f) \leq \log^c \mathrm{mon}^{\star}(f)$

Communication complexity[1] of (nontrivial) XOR and AND functions, for symmetric f:

	XOR functions	AND functions
Deterministic	$\Theta(n)$	$\Theta\left((n-t(f))\left(1+\log\frac{n}{n-t(f)}\right)\right)$
Randomized	$\Theta(r(f))$	$\Theta^{\dagger}\left((n-t(f))\left(1+\log\frac{n}{n-t(f)}\right)\right)$
Quantum	$\Theta(r(f))$	$\Theta^{\star}\left(\sqrt{n\cdot\ell_0(f)}+\ell_1(f)\right)$

[1] [ZS09, BdW01, Raz03]

Symmetric functions

Decision tree complexities[2] of (nontrivial) symmetric functions:

	Regular	Parity	Conjunctive
Deterministic	$\Theta(n)$	$\Theta(n)$	$\Theta\left((n-t(f))\left(1+\log\frac{n}{n-t(f)}\right)\right)$
Randomized	$\Theta(n)$	$\Theta(r(f))$	$\Theta^{\dagger}\left((n-t(f))\left(1+\log\frac{n}{n-t(f)}\right)\right)$
Quantum	$\Theta\left(\sqrt{n\cdot\ell(f)}\right)$	$\Theta(r(f))$	$\Theta^{\star}\left(\sqrt{n\cdot\ell_0(f)}+\ell_1(f)\right)$

[2] [ZS09, BdW01, Raz03, BBC$^+$01]

Decision tree complexities[2] of (nontrivial) symmetric functions:

	Regular	Parity	Conjunctive
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Randomized	$\Theta(n)$	$\Theta(r(f))$	$\Theta^{\dagger}\left((n-t(f))\left(1+\log\frac{n}{n-t(f)}\right)\right)$
Quantum	$\Theta\left(\sqrt{n\cdot\ell(f)}\right)$	$\Theta(r(f))$	$\Theta^{\star}\left(\sqrt{n\cdot\ell_0(f)}+\ell_1(f)\right)$

Result: Communication and Decision Tree complexities are polynomially related for symmetric functions.

[2] [ZS09, BdW01, Raz03, BBC$^+$01]

Conclusion

Our contributions:

- first efficient simultaneous protocol for $\textsc{Sym} \circ \textsc{Sym}_t$
- full characterization of the decision tree complexities of symmetric functions
- efficient construction for Ramsey numbers over $\mathbb{F}_p^n$

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Future work:

- other protocols for larger families of composed functions
- breaking the log n barrier
- log-rank conjecture for XOR and AND functions (using decision tree complexity?)

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$\textsc{Equality}$ function

$$\textsc{Equality}(x_1, \ldots, x_k) = 1 \Leftrightarrow x_1 = \cdots = x_k$$

$\mathrm{D}_2(\textsc{Equality}) = \Omega(n)$

- log-rank method

$\mathrm{R}_2^{||}(\textsc{Equality}) = \mathcal{O}(1)$

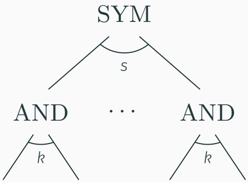
- Alice and Bob test $x \cdot r = y \cdot r \mod 2$ for two random $r \in \{0,1\}^n$

$\mathrm{D}_k^{||}(\textsc{Equality}) = \mathcal{O}(1)$ when $k > 2$

- Player 1 checks $x_2 = \cdots = x_k$
- Player 2 checks $x_1 = x_3 = \cdots = x_k$

ACC^0 and Sym$^+$

Sym$^+(s, k)$ = depth-2 circuits whose top gate is a symmetric gate of fan-in s, and each bottom gate is an AND gate of fan-in k



- $\mathrm{ACC}^0 \subset \mathrm{SYM}^+(2^{\mathrm{polylog}\, n}, \mathrm{polylog}\, n)$ [Yao90, BT94]
- f is computed by a $\mathrm{SYM}^+(s, k-1)$ circuit $\Rightarrow$ for any partition of the input between k players, there is a protocol of cost $\mathcal{O}(k \log s)$ computing f

$F(x, y) = f(x \oplus y)$ is symmetric iff f is symmetric

Symmetric XOR and AND functions

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$\rightarrow f(x)$ depends only on $|x|$. Hence $f : \{0, \ldots, n\} \rightarrow \{0, 1\}$

Symmetric XOR and AND functions

$F(x, y) = f(x \oplus y)$ is symmetric iff f is symmetric

$\rightarrow f(x)$ depends only on $|x|$. Hence $f : \{0, \ldots, n\} \rightarrow \{0, 1\}$

- $t(f) = \min\{p : f(p-1) \neq f(p)\}$
- $\ell_0(f) = \min\{p \leq n/2 : f(i) = f(n/2) \text{ for } i \in [p, n/2]\}$
- $\ell_1(f) = \min\{p \leq n/2 : f(i) = f(n/2) \text{ for } i \in [n/2, n-p]\}$
- $\ell(f) = \min\{p : f(i) = f(i+1) \text{ for } i \in [p, n-p-1]\}$
- $r(f) = \min\{p : f(i) = f(i+2) \text{ for } i \in [p, n-p-2]\}$

Ramsey numbers and $\textsc{Eval}_G$

$\textsc{Eval}_G$

For any Abelian group G and $x_1, \ldots, x_k \in G$:

$$\textsc{Eval}_G(x_1, \ldots, x_k) = 1 \Leftrightarrow x_1 + \cdots + x_k = 0$$

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Communication complexity:

- $\mathrm{R}_k^{||}(\textsc{Eval}_G) = \mathcal{O}(1)$ since

$$x_1 + \cdots + x_k = 0 \Leftrightarrow x_1 = -(x_2 \cdots + x_k)$$

- $\mathrm{D}_k(\textsc{Eval}_G) \rightarrow$ connections to Ramsey theory

k-dimensional corner in G^k:

$$(x_1, x_2, \ldots, x_k), (x_1 + \lambda, x_2, \ldots, x_k), (x_1, x_2 + \lambda, \ldots, x_k), \ldots, (x_1, x_2, \ldots, x_k + \lambda)$$

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Ramsey numbers:

- $c_k^{\angle}(G)$ = min # of colors to avoid monochromatic k-dim corner in G^k
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Chandra, Furst and Lipton [CFL83]:

$$\log(c_k^{\angle}(G)) \leq \mathrm{D}_{k+1}(\textsc{Eval}_G) \leq k + \log(c_k^{\angle}(G))$$

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Ramsey numbers and $\textsc{Eval}_{\mathbb{F}_p^n}$

Motivations for $G = \mathbb{F}_p^n$:

- the proofs are easier and cleaner
- they can be adapted to any other group [Gre05]
- $\textsc{Eval}_{\mathbb{F}_p^n} \in \textsc{Sym} \circ \textsc{Sym}_p$

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Prior work:

- $\mathrm{D}_3(\textsc{Eval}_{\mathbb{F}_p^n}) = \omega(1)$ [LM07]
- $c_k^{\angle}(\mathbb{F}_2^n) \leq \mathcal{O}\left(2^{n/2^{k-2}} n^{k+1}\right)$ [ACFN15]
- an explicit large corner free set over $\mathbb{F}_2^n$ [ACFN15]
- $c_k^{\angle}(\mathbb{F}_p^n) \leq 2^{\mathcal{O}\left(p \log^2 n\right)} p^{\mathcal{O}(p \log n)}$ when $k > 1 + p \log(3n)$ [CS14]

Ramsey numbers and $\text{EVAL}_{\mathbb{F}_p^n}$

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Our result:

- the first explicit large corner-free set over $\mathbb{F}_p^n$, of size $\frac{p^{nk}}{C^{k^2} p^{k+k^2}}$

Our contribution: the first explicit large corner-free set in $\mathbb{F}_p^n$

Results

Our contribution: the first explicit large corner-free set in $\mathbb{F}_p^n$

Definitions:

- $M \in (\mathbb{F}_p^n)^k$ is seen as a $k \times n$ matrix over $\mathbb{F}_p$
- $d(c, c_j) =$ Hamming distance between columns c and c_j
- $n_{i,c}(M) =$ number of columns at distance i to c in M

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For any $c \in \mathbb{F}_p^k$, $N_k = 0$ and $N_0, \ldots, N_{k-1} \geq 0$ such that $\sum_{i=0}^{k} N_i = n$:

$$S_c^k = \{M \in (\mathbb{F}_p^n)^k : \forall i \in \{0, \ldots, k\}, n_{i,c}(M) = N_i\}$$

is a corner-free set.

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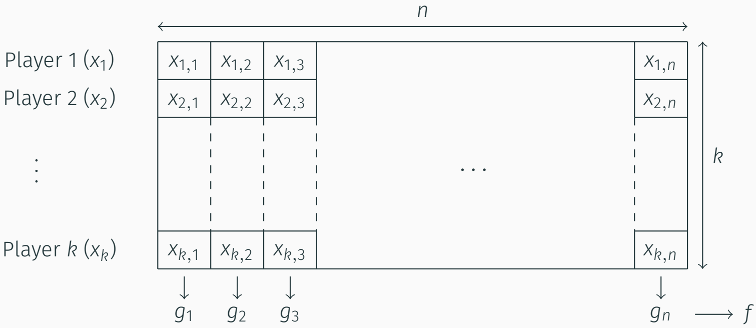
is a corner-free set.

If $k \geq \left\lceil \frac{\log n}{\log\left(1+\frac{1}{p-1}\right)} \right\rceil$ and $N_i = \left\lfloor \binom{k}{i} \frac{(p-1)^i}{p^k} n \right\rfloor$ then $|S_c^k| \geq \frac{p^{nk}}{C^{k^2} p^{k+k^2}}$

The $\log n$ barrier and composed functions

Given $f : \{0,1\}^n \to \{0,1\}$ and $\overrightarrow{g} = (g_1, \ldots, g_n)$ where $g_i : \{0,1\}^k \to \{0,1\}$:

$$f \circ \overrightarrow{g}(x_1, \ldots, x_k) = f(\ldots, g_i(x_{1,i}, \ldots, x_{k,i}), \ldots)$$



Composed functions

Definitions:

- $f \circ g$ if $g_1 = \cdots = g_n$
- Symmetric = invariant under any permutation of the input
- $\textsc{Any} \circ \overrightarrow{\textsc{Any}}$, $\textsc{Any} \circ \textsc{Any}$, $\textsc{Sym} \circ \overrightarrow{\textsc{Any}}$, $\textsc{Sym} \circ \textsc{Sym}$...

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Motivations:

- very simple structure
- most of the important functions: $\mathrm{GIP} = \mathrm{MOD}_2 \circ \mathrm{AND} \in \textsc{Sym} \circ \textsc{Sym}$, $\mathrm{MAJ} \circ \mathrm{MAJ} \in \textsc{Sym} \circ \textsc{Sym}$, $\mathrm{DISJ} = \mathrm{NOR} \circ \mathrm{AND} \in \textsc{Sym} \circ \textsc{Sym}$
- major open problems still unknown for composed functions

Prior work

Conjecture ([BKL95]): $\mathrm{MAJ} \circ \mathrm{MAJ}$ breaks the $\log n$ barrier

Prior work

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When $k = \Omega(\log n)$:

- $\mathrm{D}_k(f \circ g) = \mathcal{O}\left(\log^2 n\right)$ for $f \circ g \in \textsc{Sym} \circ \mathrm{AND}$ [Gro94]
- $\mathrm{D}_k^{||}(f \circ g) = \mathcal{O}\left(\log^3 n\right)$ for $f \circ g \in \textsc{Sym} \circ \textsc{Comp}$ [BGKL04]
- $\mathrm{D}_k^{||}(f \circ \overrightarrow{g}) = \mathcal{O}\left(\log^3 n\right)$ for $f \circ \overrightarrow{g} \in \textsc{Sym} \circ \overrightarrow{\textsc{Any}}$ [ACFN15]

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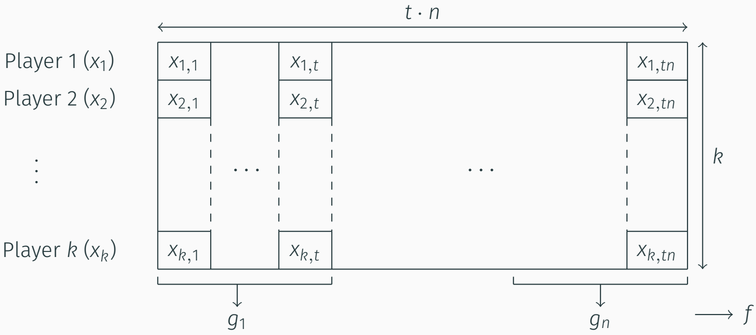
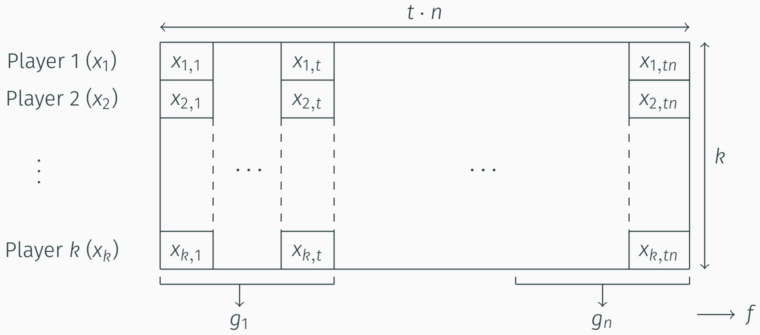
- $\mathsf{D}_k^{||}(f \circ \vec{g}) = \mathcal{O}\left(\log^3 n\right)$ for $f \circ \vec{g} \in \textsc{Sym} \circ \overrightarrow{\textsc{Any}}$ [ACFN15]

$\rightarrow$ none of the functions in $\textsc{Sym} \circ \overrightarrow{\textsc{Any}}$ can break the $\log n$ barrier

Composed functions of block-width t

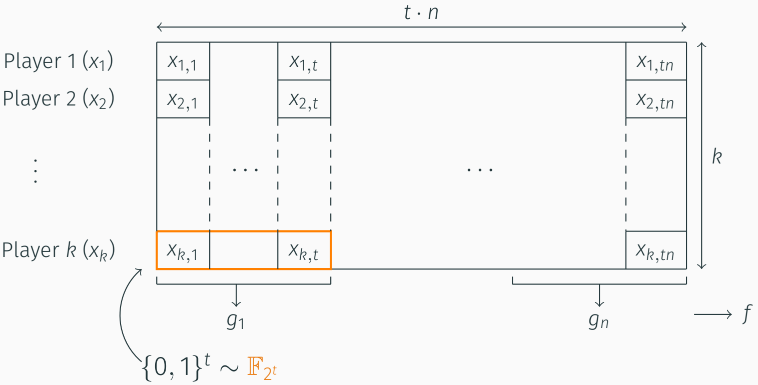


Composed functions of block-width t



- $\mathrm{MAJ}_t : \{0,1\}^{k \cdot t} \to \{0,1\}$
- Conjecture : $\mathrm{MAJ} \circ \mathrm{MAJ}_{\sqrt{n}}$ breaks the barrier

Composed functions of block-width t



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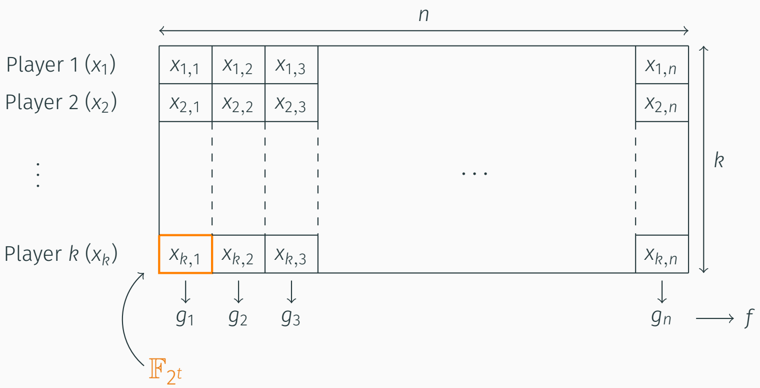

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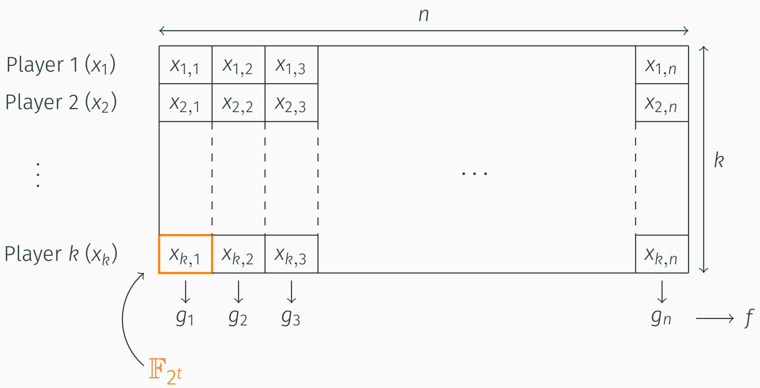

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$$f \circ \overrightarrow{g}(x_1, \ldots, x_k) = f(\ldots, g_i(x_{1,i}, \ldots, x_{k,i}), \ldots)$$

$\rightarrow$ $\textsc{Any} \circ \overrightarrow{\textsc{Any}_p}, \textsc{Any} \circ \textsc{Any}_p, \textsc{Sym} \circ \textsc{Any}_p, \ldots$

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Conjecture: $\mathrm{MAJ} \circ \mathrm{MAJ}_{\sqrt{\log n}}$ breaks the $\log n$ barrier

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When $k = \Omega(\operatorname{polylog} n)$:

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- $\mathrm{D}_k(f \circ g) = \mathcal{O}\left(polylogn\right)$ for $f \circ g \in \textsc{Sym} \circ \overrightarrow{\textsc{Any}_p}$ and $p \leq \operatorname{polylog} n$ [CS14]

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New results for constant p:

- $D_k^{||}(f \circ g) = \mathcal{O}\,(polylogn)$ for $f \circ g \in \textsc{Sym} \circ \textsc{Sym}_p$ $(k = \operatorname{polylog} n)$
- $D_k^{||}(f \circ g) = \mathcal{O}\,(polylogn)$ for $f \circ g \in \textsc{Sym} \circ \textsc{Comp}_p$ $(k \geq \operatorname{polylog} n)$
- $\mathrm{MAJ} \circ \mathrm{MAJ}_t$ cannot break the barrier for constant t

Proof sketch

Symmetric f and g over $\mathbb{F}_3$:

0	0	1	2	0	2	3	2	1	1
1	0	3	1	0	1	1	0	2	0
0	0	3	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
3	0	0	3	1	0	1	0	2	0

$g \quad g \quad \cdots \quad g \longrightarrow f$

$y_{i,j}$ = # columns with i one's and j two's

Proof sketch

Symmetric f and g over $\mathbb{F}_3$:

0	0	1	2	0	2	3	2	1	1
1	0	3	1	0	1	1	0	2	0
0	0	3	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
3	0	0	3	1	0	1	0	2	0

$g \quad g \quad \cdots \quad g \longrightarrow f$

$y_{i,j}$ = # columns with i one's and j two's

$\rightarrow y_{0,0} = 1$

Proof sketch

Symmetric f and g over $\mathbb{F}_3$:

0	0	1	2	0	2	3	2	1	1
1	0	3	1	0	1	1	0	2	0
0	0	3	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
3	0	0	3	1	0	1	0	2	0

$g \quad g \quad \cdots \quad g \longrightarrow f$

$y_{i,j} =$ # columns with i one's and j two's

$\rightarrow y_{0,0} = 1, y_{1,0} = 2$

Proof sketch

Symmetric f and g over $\mathbb{F}_3$:

0	0	1	2	0	2	3	2	1	1
1	0	3	1	0	1	1	0	2	0
0	0	3	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
3	0	0	3	1	0	1	0	2	0

$\downarrow$ $\downarrow$ $\ldots$ $\downarrow$

g g $g \longrightarrow f$

$y_{i,j}$ = # columns with i one's and j two's

$\to y_{0,0} = 1, y_{1,0} = 2, y_{1,1} = 1$

Proof sketch

Symmetric f and g over $\mathbb{F}_3$:

0	0	1	2	0	2	3	2	1	1
1	0	3	1	0	1	1	0	2	0
0	0	3	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
3	0	0	3	1	0	1	0	2	0

$g \quad g \quad \cdots \quad g \longrightarrow f$

$y_{i,j}$ = # columns with i one's and j two's
$\rightarrow y_{0,0} = 1, y_{1,0} = 2, y_{1,1} = 1, \ldots$

$\boxed{\text{Recovering the } y_{i,j}\text{'s is enough since } f \text{ and } g \text{ are symmetric}}$

Proof sketch

0	0	1	2	0	2	3	2	1	1
1	0	3	1	0	1	1	0	2	0
0	0	3	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
3	0	0	3	1	0	1	0	2	0

- Player 1 sends to the referee:

$$a^1_{i,j} = \#\text{ columns she sees with } i \text{ one's and } j \text{ two's}$$

$\rightarrow a^1_{0,0} = 2, a^1_{1,0} = 1, a^1_{1,1} = 3, \ldots$

Proof sketch

0	0	1	2	0	2	3	2	1	1
1	0	3	1	0	1	1	0	2	0
0	0	3	2	0	0	1	2	1	0
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- Players 2 to 5 do the same

Proof sketch

0	0	1	2	0	2	3	2	1	1
1	0	3	1	0	1	1	0	2	0
0	0	3	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
3	0	0	3	1	0	1	0	2	0

The referee computes:

$$b_{i,j} = a_{i,j}^1 + \cdots + a_{i,j}^5$$

Proof sketch

0	0	1	2	0	2	3	2	1	1
1	0	3	1	0	1	1	0	2	0
0	0	3	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
3	0	0	3	1	0	1	0	2	0

The referee computes:

$$b_{i,j} = a^1_{i,j} + \cdots + a^5_{i,j}$$

Note that:

- $b_{0,0} =$

Proof sketch

0	0	1	2	0	2	2	2	1	1
1	0	0	1	0	1	1	0	2	0
0	0	1	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
2	0	0	1	1	0	1	0	2	0

The referee computes:

$$b_{i,j} = a^1_{i,j} + \cdots + a^5_{i,j}$$

Note that:

- $b_{0,0} = 5y_{0,0}$

Proof sketch

0	0	1	2	0	2	2	2	1	1
1	0	0	1	0	1	1	0	2	0
0	0	1	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
2	0	0	1	1	0	1	0	2	0

The referee computes:

$$b_{i,j} = a^1_{i,j} + \cdots + a^5_{i,j}$$

Note that:

- $b_{0,0} = 5y_{0,0} + y_{1,0}$

Proof sketch

0	0	1	2	0	2	3	2	1	1
1	0	3	1	0	1	1	0	2	0
0	0	3	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
3	0	0	3	1	0	1	0	2	0

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1	0	0	1	0	1	1	0	2	0
0	0	1	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
2	0	0	1	1	0	1	0	2	0

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1	0	0	1	0	1	1	0	2	0
0	0	1	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
2	0	0	1	1	0	1	0	2	0

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0	0	3	2	0	0	1	2	1	0
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1	0	3	1	0	1	1	0	2	0
0	0	3	2	0	0	1	2	1	0
0	0	2	1	0	1	2	1	2	0
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- $\ldots$

$$b_{i,j} = (k - (i + j))y_{i,j} + (i + 1)y_{i+1,j} + (j + 1)y_{i,j+1}$$

Proof sketch

Let $(b_{i_1,\ldots,i_p})_{0\leq i_1+\cdots+i_p\leq k-1}$ be integers. Consider the system of equations:

$$\begin{cases} (k-(i_1+\cdots+i_p))y_{i_1,\ldots,i_p} + \sum_{j=1}^{p}(i_j+1)y_{i_1,\ldots,i_{j-1},i_j+1,i_{j+1},\ldots,i_p} = b_{i_1,\ldots,i_p} \\ 0 \leq i_1+\cdots+i_p \leq k-1 \end{cases}$$

Assume further that

$$y_{i_1,\ldots,i_p} \geq 0,\ 0 \leq i_1+\cdots+i_p \leq k \quad \text{and} \quad \sum_{i_1+\cdots+i_p\leq k} y_{i_1,\ldots,i_p} \leq n$$

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Theorem

If $k > 1+5^p \log n$ then it admits at most one integral solution.

Proof sketch

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Theorem

If $k > 1 + 5^p \log n$ then it admits at most one integral solution.

$\rightarrow$ the referee recovers the $y_{i,j}$'s and computes the output

Proof sketch

Conclusion:

- [BGKL04] proved the uniqueness for $p = 2$
- we generalized to any p
- sending all the $a_{i,j}^{\ell}$ has cost $\mathcal{O}\left(k(k+p)\log n\right) \to$ not efficient is $k = \omega(\operatorname{polylog} n)$ (compressibility)

Proof sketch

Conclusion:

- [BGKL04] proved the uniqueness for $p = 2$
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Future work:

- remove the compressibility condition
- handle larger p